INTRODUCTION
TO
RESURGENCE
&
ASYMPTOTICS

- WORKING NOTES, EXAMPLES MIGHT CHANGE!

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ASYMPTOTICS IN THEORETICAL / MATHEMATICAL PHYSICS

- Black holes, conformal blocks & SW potentials
  ↳ Exact quantization techniques, Painlevé eqs, –
  (Bonelli, Grossi, Bianchi,
- q-series, quantum Modularity
- Correlation functions / Free energies in context of topological strings (relation topological recursion) matrix models (including 2D gravity, JT gravity), integrable field theories / TBA
- Relativistic hydrodynamics and fluid/gravity correspondence (energy density of strongly coupled fluids)
- More on the Math phys applications: hydrodynamics singular behaviour in IVPs / singularly perturbed problems, BVPs, ...
  Integrable Models such as KdV hierarchy
- Pure maths applications – NLODEs / PDEs
- Connections with string amplitudes

- In many physical applications, one often needs to treat our observeble perturbatively
  - QFTs weak/strong coupling
  - Black holes large charge
  - ℏ expansions in QM ...

⇒ Obtain expansions which are often (not always) asymptotic divergent.

- Behind asymptotic behaviour: non-perturbative phenomena, zero at all orders in perturbation theory which needs to somehow be incorporated to have a full picture of our observable beyond its local perturbative behaviour.

- The question is then: how useful are these divergent series? Or in other words how can we retrieve information from them?

→ Asymptotic series can be much more effective at approximating function: Exponential accuracy
→ In the large order divergent behaviour they encode information on n.p. physics in quite explicit ways: RESURGENCE
→ One can include perturbative & n.p. contributions to a particular observable constructing its n.p. completion in the form of a so-called Transseries
→ There are different very effective summation methods which allows us to obtain a meaningful number from asymptotic perturbative expansion and the corresponding Transseries.

1. RESURGENCE & STOKES PHENOMENON

1.1. The accuracy of asymptotic series
  - Convergence vs divergence
  - Transseries as a general solution?

1.2. Universality of large order behaviour & Resurgence
  - Optimal truncation & exact remainders
  - Large order behaviour beyond the leading term

1.3. The emergence of Stokes Phenomena
  - Saddle point approximation, complex saddles & transseri
  - Borel transforms, Resurgence & large order behaviour
  - The physics at Stokes lines & anti-stokes lines

1.4. A Glimpse towards exponential accuracy
  - Optimal truncation
  - Borel sumation

2. ECALLE'S RESURGENCE & EXPONENTIAL ASYMPTOTICS

2.1 RESURGENT TRANSSERIES as Global solution
- Borel analysis & singularity structure
- Stokes automorphism
- Monodromy

2.2. Borel analysis as a method of constructing transseries
- Borel - Padé
- Large order analysis

2.3. Ambiguity cancellation & Borel summability
- Real solution

3. SUMMATION & EXPONENTIAL ACCURACY

3.1. Beyond optimal truncation
  - Opt. truncation revisited
  - hyperterminants & level-1 hyperasymptotics

3.2. Borel summation
  - Borel-padé
  - Conformal maps

3.3. Transasymptotics

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1. Stokes Phenomenon :

1.1 In the late 1800s (1886) a first definition of asymptotic series came to be: Poincaré asymptotics (applied to both convergent and divergent series)

For $y(z): \mathbb{C} \to \mathbb{C}$ and $z \in S \subset \mathbb{C}$ particular sector in complex plane

$$y(z) \underset{\uparrow}{\sim} \sum_{n=0}^{+\infty} a_n z^{-n} \qquad |z| \to \infty$$

means $\forall N \geq 1$ remainder $R_N(z)$ is such that

$$R_N(z) = y(z) - \sum_{n=0}^{N-1} a_n z^{-n} = O(z^{-N}) \leq K |z|^{-N}$$

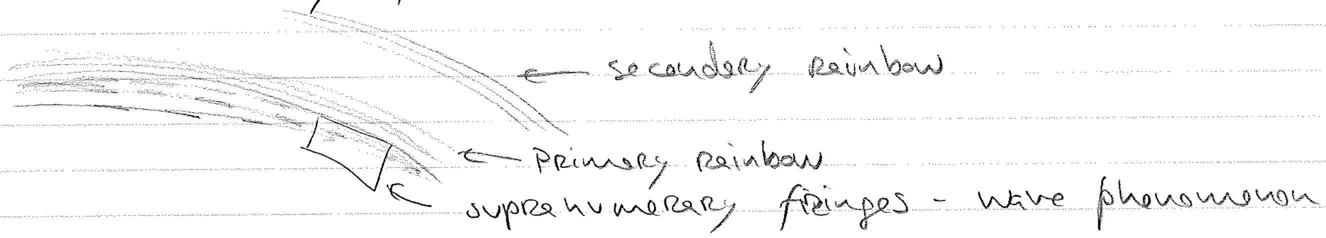
But neglects any terms $O(e^{-z})$, same asymptotics

$$y(z) + e^{-z} \sim \sum_{n=0}^{\infty} a_n z^{-n} + \sum_{n=0}^{\infty} 0 \, z^{-n} \sim y(z)$$

As we will see with Stokes phenomena, there will be regions $S' \neq S$ where terms proportional to e^{-z} will appear and can then grow to become large! How to incorporate it in the asymptotics?

Let us go back to one of the first problems with Stokes phenomena : The Airy function

George B. Airy (~ 1838) became interested in the so called supranumerary fringes of rainbows (less obvious arcs, mostly green pink and purple, beneath the main body of the main rainbow)

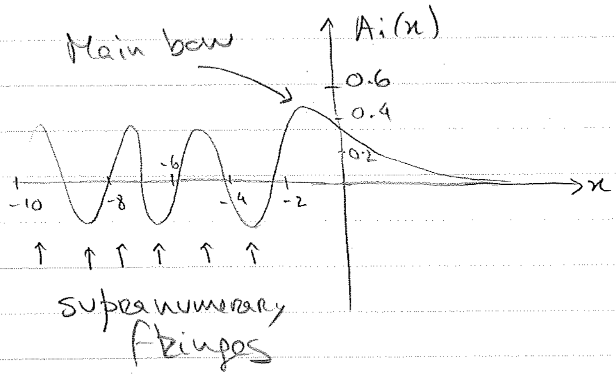


The reason for this interest was because a similar fringing effect occurs in optical lenses. Airy was an astronomer and he wanted to understand ~~the~~ theory behing this phenomenon, to improve optics of telescopes.

<u>History</u>: early 17th century René Descarte explained origin of reimbow using theory of rays of light but this did not predict the fringes.
Airy used wave theory of light and wrote a mathematical formula known as the Airy function for which the intensity of light of primary & secondary rainbows can be obtained.
Along the axis perpendicular to the rainbow, the intensity of light is given by square of Airy function:

Intensity of wave $y^2(x)$ $y(x) = Ai(x)$ obeys

$$\frac{d^2y}{dx^2} - xy = 0 \qquad \underbrace{y(0) = \frac{1}{3^{2/3}\Gamma(2/3)} \qquad y'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}}_{\text{Initial data}}$$

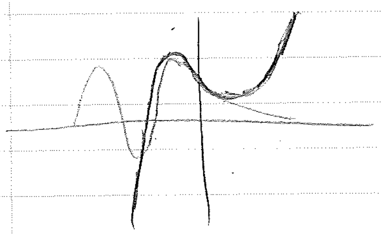


Airy wanted to predicts the fringes.

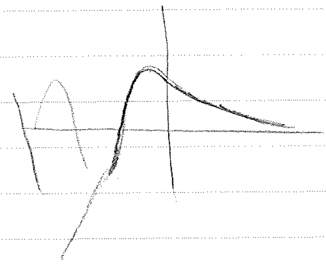
→ Approximate solution by a series of terms around the origin

Convergent series: calculating many terms would be very time consuming and the convergence was just too slow:

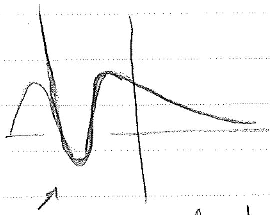
4 Terms



12 terms



19 terms

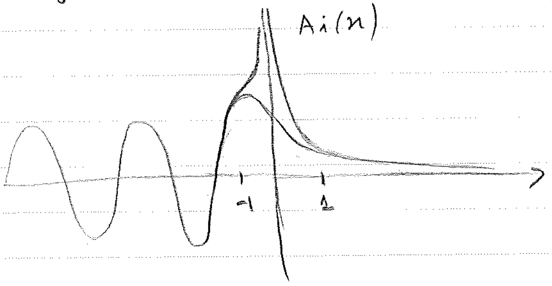


to obtain first finite minimum

It was in 1850 George G. Stokes came up with the use of divergent series for approximation. He saw that performing an analysis for large x

$$Ai(x) \sim \begin{cases} e^{-2/3\, x^{3/2}} / 2\sqrt{\pi}\, x^{1/4} & x > 0 \quad \leftarrow \text{one exponential} \\ \dfrac{\sin\left(2/3\,(-x)^{3/2} + \pi/4\right)}{\sqrt{\pi}\,(-x)^{1/4}} & x < 0 \quad \leftarrow \text{two exponentials} \end{cases}$$

These are just the leading terms in an expansion $|x| \to \infty$ and they give a remarkably good approximation appart from a region close to the origin:



But obviously the asymptotic expansion valid for $x > 0$ will be valid in a sector $S \subset \mathbb{C}$ which includes the $\mathbb{R}^+$ but cannot include $\mathbb{R}^-$, as in $\mathbb{R}^-$ the asymptotic behaviour clearly includes 2 exponentials!

How can we explain this? Stokes phenomenon

$\boxed{1.1.2}$ Look back at the Airy equation

$$\frac{d^2y}{dx^2} - x\,y = 0$$

formal large x solution

$$y(x) \approx \frac{e^{-\rho(x)}}{\sqrt{\rho'(x)}}\,(1 + \dots) \qquad \left(\rho(x) - \text{Écalle variable}\right)$$

Two independent solutions

$$y_1(x) = Ai(x) \simeq \frac{e^{-\rho}}{2\sqrt{\pi}\, x^{1/4}} \sum_{n \geq 0}^{\infty} (-1)^n c_n \rho^{-n} \qquad -\pi < \arg x < \pi$$

$$y_2(x) = e^{-\pi i/6} Ai\left(x e^{-2\pi i/3}\right) \simeq \frac{e^{\rho}}{2\sqrt{\pi}\, x^{1/4}} \sum_{n \geq 0} c_n \rho^{-n} \qquad -\pi/3 < \arg x < \frac{5\pi}{3}$$

These are formal series expansions; <u>divergent</u>

C_n can be determined via recursion relations (Exercise)

$$C_n = \frac{\Gamma(n+1/6)\,\Gamma(n+5/6)}{2\pi\, n!\, 2^n} \qquad (C_0 = 1)$$

Most General solution : <u>Transseries</u>

$$y(x) = \sigma_1 y_1(x) + \sigma_2 y_2(x)$$

$\uparrow$

generally it can have two exponentials and power series in ξ^{-1}

ASIDE : Transseries is formal power series in different monomials, each representing a different scale in the observable

<u>Example</u>
Energies in QM

$$\sum_{n,m,k} e^{-nA/\hbar} \sum_m \ln^m \hbar \sum_k C^{(n)}_{k,m} \hbar^k$$

trans-monomials $e^{-A/\hbar}$, $\ln \hbar$, $\hbar$

transseries are generally suficient to discribe non-pert completion of physical observables, using those trans-monom

The $C^{(n)}_{k,m}$ are in fact intimately related to each other for different n, n' : <u>Resurgence</u>

Along $\arg x = 0$ $\quad y = y_1(x) \sim \dfrac{e^{-\xi}}{2\sqrt{\pi}\, x^{1/4}} \Rightarrow C_2 = 0,\ C_1 = 1$

Along $\arg x = \pi$ $\quad y \sim \dfrac{e^{i|\xi| - 1/4\pi i}}{2\sqrt{\pi}\, |x|^{1/4}} + i\, \dfrac{e^{-i|\xi| - 1/4\pi i}}{2\sqrt{\pi}\, |x|^{1/4}}$

$$\Rightarrow \sigma_1 = 1,\ \sigma_2 = i$$

Coefficient σ_2 changes its value!

In order to understand this change which brings an extra exponential we need to understand the behaviour of y in the whole complex plane.

$\boxed{1.2}$ Let us analyse the asymptotic series $|x| \to \infty$

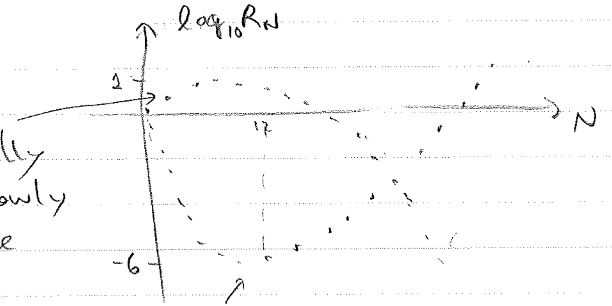
$$y_1(x) = \frac{e^{-\rho}}{2\sqrt{\pi}\, x^{1/4}} \left(\sum_{n=0}^{N-1} \frac{c_n (-1)^n}{\rho^n} + R_N^{(a)} \right) \qquad \rho = \tfrac{2}{3} x^{3/2}$$

→ Truncate the series and analyse the remainder $R_N^{(a)}$

Compare with Taylor expansion around $x=0$ (convergent)

$$A_i(x) = \sum_{n=0}^{N-1} c_n x^n + R_N^{(c)}$$

At $\underline{x = 4}$



error initially increases, then slowly begins to converge

Error decreases up to least error then gets worse

<u>OPTIMAL TRUNCATION</u> : truncation to least error

$$y_1 \sim \sum_{n \geq 0} a_n \qquad a_n = \frac{c_n}{\rho^n_n} \frac{e^{-\rho}}{2\sqrt{\pi}\, x^{1/4}}$$

We want $\left| \dfrac{a_n}{a_{n+1}} \right| \gtrsim 1 \Rightarrow \dfrac{(n-1/6)(n-5/6)}{2n|\rho|} \underset{n\to\infty}{\sim} \dfrac{n}{2|\rho|} \gtrsim 1$

Then $n_{op} = \lfloor 2|\rho| \rfloor$ (2 is related to 2ρ being distance between exponentials)

Size of least term $|a_n| = \left| \dfrac{e^{-\rho}}{2\sqrt{\pi}\, x^{1/4}} \dfrac{c_n}{\rho^n} \right|$ where $n = 2|\rho|$?

Assuming $|\rho| \to \infty$ $n_{op} \gg 1$

Stirling approximation $\Gamma(n+\mu) \sim \sqrt{2\pi}\, n^{n+\mu-1/2} e^{-n}$

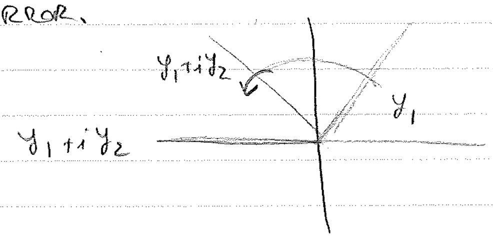
$$|a_n| \sim \left| \frac{e^{-\rho}}{2\sqrt{\pi}\, x^{1/4}} \frac{\Gamma(n+0)}{2\pi (2\rho)^n} \right| \sim \frac{e^{-\rho-2|\rho|} (2|\rho|)^{2|\rho|-1/2} \sqrt{2\pi}}{4\pi^{3/2} x^{1/4} (2\rho)^{2|\rho|}} = \frac{\sqrt{3/}}{\pi |x|} e^{-\rho-2|\rho|}$$

The least term is exp small and smaller then the second exponential we would be adding by turning on $y_2 \sim |e^{\rho}|$, except if $\arg \rho = \pi \Rightarrow \arg x = 2/3\, \pi$

$$e^{\rho} = e^{-|\rho|} \qquad e^{-\rho-2|\rho|} \neq e^{-|\rho|} \quad //$$

At this point error of opt truncation and new exponential balance each other and adding the second solution will not create a discrete jump in error.

Stokes line



Universality of large order behaviour

Goes back to Dingle in the 1950's

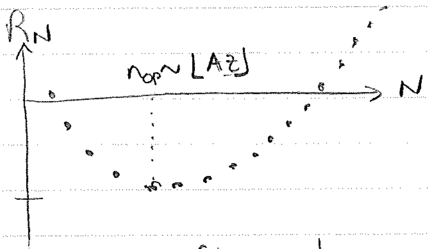
→ The late terms of asymptotic series have universal behaviour

Resurgence

→ The divergence of Perturbative series and Stokes phenomenon (whereas an exponentially small contribution gets picked up) are intimately linked

→ Diverging late terms / large orders encode information about other solutions / saddles / non-pert phenomena.

→ Made use of Darboux's method & Borel summation



$$y(z) \sim \sum_{n \geq 0} a_n z^{-n} \qquad |z| \to \infty$$

$$z^{-n} a_n \sim K \frac{\Gamma(n+\mu)}{(zA)^{n+\mu}} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \quad n \to \infty$$

→ coefficients behave as factorial over power

→ exponential accuracy by sum up to least term

$$\left|z^{-n_{op}} a_{n_{op}}\right| \sim K \sqrt{2\pi} \frac{e^{-A|z|}}{\sqrt{A|z|}} \qquad \left(\Gamma(n+\mu) \simeq \sqrt{2\pi}\, n^{n+\mu-1/2} e^{-n}\right)$$

⇒ Remainder $\quad y(z) = \sum_{n=0}^{N-1} a_n z^{-n} + R_N(z, A)$

Where $R_{N_{op} \sim A|z|} = \mathcal{O}\left(e^{-A|z|} / \sqrt{A|z|}\right)$

▶ But we can go beyond the leading term is large order of $z^{-n} a_n$ for large $n \to \infty$.

$$a_n \sim K \frac{\Gamma(n+\mu)}{A^{n+\mu}} \left(1 + \frac{d_1}{n} + \frac{d_2}{n^2} + \cdots\right)$$

What information do c_1, c_2 encode?

- Look back at Airy solutions y_1, y_2

$$y_1 \simeq \frac{e^{-\xi}}{2\sqrt{\pi}\,x^{\frac{1}{4}}} \sum_{n\geq 0}^{\infty} \frac{c_n(-1)^n}{\xi^n} \qquad y_2 \sim \frac{e^{\xi}}{2\sqrt{\pi}\,x^{1/4}} \sum_{n\geq 0} c_n \xi^{-n}$$

→ Use Stirlings approximation (DLMF 5.11.3)

$$\Gamma(z) = e^{-z} z^z \left(\frac{2\pi}{z}\right)^{1/2} \sum_{k=0}^{\infty} \frac{g_k}{z^k}$$

The first few g_k $g_0 = 1$, $g_1 = 1/12$, $g_2 = \frac{1}{288}$, $g_3 = -\frac{139}{51840}$

<u>Large order from y_1</u>:

$$y_1: \quad \frac{c_n(-1)^n}{\xi^n} = \frac{1}{2\pi} \frac{\Gamma(n+1/6)\Gamma(n+5/6)}{n!\,(-4/3\,x^{3/2})^n} \sim \frac{\Gamma(n)}{2\pi(-\frac{4}{3}x^{3/2})^n}\left(1 - \frac{5}{36}\frac{1}{n} + \frac{25}{2592}\frac{1}{n^2} + \frac{775}{279936}\frac{1}{n^3}\right)$$

$$\underset{n\to\infty}{\simeq} \frac{1}{2\pi}\frac{\Gamma(n)}{(-\frac{4}{3}x^{3/2})^n}\left[1 + \left(\frac{-4}{3}\right)\underline{\underline{\left(\frac{5}{48}\right)}}\frac{1}{n-1} + \left(\frac{-4}{3}\right)^2\underline{\underline{\left(\frac{385}{4608}\right)}}\frac{1}{(n-1)(n-2)} + \right.$$
$$\left. + \left(-\frac{4}{3}\right)^3 \underline{\underline{\left(\frac{85085}{663552}\right)}}\frac{1}{(n-1)(n-2)(n-3)} + \cdots\right]$$

Coefficients of y_2

$$y_2: \quad \frac{c_n}{\xi^n} = \frac{1}{2\pi}\frac{\Gamma(n+1/6)\Gamma(n+5/6)}{n!\,(4/3\,x^{3/2})^n} = \frac{1}{x^{3n/2}}\left\{1, \frac{5}{48}, \frac{385}{4608}, \frac{85085}{663552}, \dots\right\}$$

⇒ In the larger order of y_1 we find the low order coefficients of the second solution y_2!

Large order relation, where the coefficients of y_2 <u>RESURGE</u> in the large order behaviour of the coefficients of y_1!

- In other words the fluctuation series about two saddles (as we will see) are explicitely related, one encodes information about the other

- The leading factorial over power law behaviour also provides further information

$$\frac{c_n(\mp)^n}{\xi^n} \sim \frac{1}{2\pi(\mp\frac{4}{3}x^{3/2})^n} \simeq \frac{\Gamma(n)}{\underbrace{(\mp A x^{3/2})^n}} \, \frac{\textcircled{i}}{2\pi i} \leftarrow \text{Stokes multiplier}$$

Difference between exponential weights of y_1, y_2!

$S_1 = i$ is the Stokes multiplier, which we saw earlier was the jump of C_2 from $\mathbb{R}^+$ to $\mathbb{R}^-$

$\mathbb{R}^+$ $\quad Ai(x) \sim y_1$

$\mathbb{R}^-$ $\quad Ai(x) \sim y_1 + \textcircled{i}\, y_2$ //

1.3 Emergence of Stokes Phenomenon

There is a way of studying the jumps needed to explain 1 exp → 2 exp of the Airy solution, and to derive the encoding we just saw in the large order behaviour:

1.3.1 Borel summation

Assume we have $\Phi(g) \simeq \sum_{n\geq 0} \frac{c_n(-1)^n}{g^{n+\beta}}$

(In y_1 of $Ai(x)$ we have $y_1 \sim \frac{e^{-g}}{2\sqrt{\pi}\, x^{1/4}} g^{\beta}\Phi$)

For $n \gg 1$ $\quad c_n \sim \frac{\Gamma(n+\beta)}{A^{n+\beta}} \frac{S_1}{2\pi i}\left(1 + O\left(\frac{1}{n}\right)\right)$ $\quad$ (Airy $\beta = 0$)

Use integral representation of $\Gamma(n+\beta)$ (DLMF 5.9)

$$\Gamma(\alpha) z^{-\alpha} = \int_0^\infty dt\; e^{-zt}\, t^{\alpha-1}$$

Then
$$\frac{c_n(-1)^n}{g^{n+\beta}} = \underbrace{\frac{c_n(-1)^n}{\Gamma(n+\beta)}}_{\text{Removes factorial growth of } c_n!} \int_0^\infty dt\, e^{-gt}\, t^{n+\beta-1}$$

→ Interchange summation in Φ / integration

$$\sum_{n\geq 0} \frac{c_n(-1)^n}{g^{n+\beta}} = \sum_{n\geq 0} \frac{c_n(-1)^n}{\Gamma(n+\beta)} \int_0^\infty dt\, e^{-gt}\, t^{n+\beta-1}$$

$$\left(S\Phi(g) :=\right) \int_0^\infty dt\, e^{-gt} \underbrace{\boxed{\sum_{n\geq 0} \frac{c_n(-1)^n}{\Gamma(n+\beta)} t^{n+\beta-1}}}_{\text{convergent sum}} \quad \underline{B[\Phi](t)} \quad \underline{\text{Borel transform}}$$

Define above integral <u>Borel Sum</u> $\underline{S\Phi(g)}$

Example Euler series

$$\sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^{n+1}} = \Phi(x)$$

$$S\Phi(x) = \int_0^{\infty} dt\, e^{-xt}\, B[\Phi](t) \quad , \quad B[\Phi](t) = \sum_{n\geq 0} (-1)^n t^{n+1-1} = \frac{1}{1+t} \;//$$

$\downarrow$

(Integral representation of incomplete Gamma function)
(Directions with singularities Borel plane $\leftrightarrow$ connection formulas)

Back to Airy function

$$y_1 = \frac{e^{-\rho}}{2\sqrt{\pi}\, x^{1/4}} \Phi_1(x^{3/2}) \qquad \Phi_1(\rho) = \sum_{n\geq 0} \frac{c_n(-1)^n}{(4/3)^n} x^{-3n/2} \qquad (\text{Take } z = x^{3/2})$$

$$B[\check{\Phi}_1](t) = \sum_{n\geq 1} \frac{c_n(-1)^n t^{n-1}}{\Gamma(n)\,(4/3)^n} = \sum_{n\geq 1} \frac{1}{2\pi} \frac{\Gamma(n+1/6)\,\Gamma(n+5/6)}{n!\,\Gamma(n)\,t} \left(\frac{-3t}{4}\right)^n$$

Note: we left out the constant term in $\Phi_1 = 1 + \check{\Phi}_1$ as the constant term cannot be transformed

$$B[\check{\Phi}_1](t) = \frac{1}{2\pi}\left(-\frac{3}{4}\right) \sum_{n\geq 0} \frac{\Gamma(n+7/6)\,\Gamma(n+11/6)}{\Gamma(n+2)\,n!} \left(-\frac{3}{4}t\right)^n$$

$$= \frac{1}{2\pi}\left(-\frac{3}{4}\right) \frac{\Gamma(7/6)\,\Gamma(11/6)}{\Gamma(2)}\; {}_2F_1\left(7/6, 11/6; 2; -3/4\, t\right)$$

$$= -\frac{5}{48}\; {}_2F_1\left(7/6, 11/6; 2; -t/A\right)$$

$A = 4/3$ | distance between two exponentials

Exercise: show $B[\check{\Phi}_2](t) = +\frac{5}{48}\; {}_2F_1\left(7/6, 11/6; 2; t/A\right)$

$$y_2 = \frac{e^{\rho}}{2\sqrt{\pi}\, x^{1/4}} \Phi_2(x^{3/2})$$

1.3.2 Borel Summation and analytic continuation

Knowing $B[\Phi_1]$ & $B[\Phi_2]$ the solution to Airy equation is given by the Borel sum of transseries

$$S_\theta y = \sigma_1 \frac{e^{-\rho}}{2\sqrt{\pi}\, x^{1/4}} S_\theta \Phi_1 + \sigma_2 \frac{e^{\rho}}{2\sqrt{\pi}\, x^{1/4}} S_\theta \Phi_2$$

This sum is not restricted to the real axis

$$S_\theta \Phi_1 = \int_0^{e^{i\theta}\infty} dt\, e^{-zt} B[\Phi_1](t) \qquad z = x^{3/2}$$

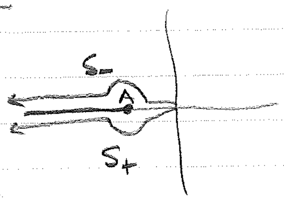
This provides an analytic continuation of the summed result to any $z \in \mathbb{C}$ as long as we take the Borel sum such that $\mathrm{Re}(zt) > 0$ to guarantee convergence.

However $B[\Phi_1](t)$ has singularities in the complex t-plane

$B[\Phi_1](t)$ has a log-cut singularity starting at $t=-A$ along negative real axis

Thus for $\theta = \pi$ one cannot define the integration, and needs to define Lateral Borel sums

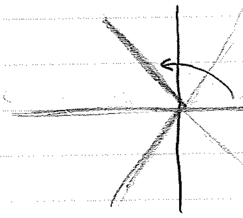
▶ $S_{\theta\pm}\Phi(z) = \int_0^{e^{i\theta\pm i\epsilon}\infty} dt\, e^{-zt} B[\Phi](t)$



(*) In order to find the analytic continuation for $S\Phi(z)$ $z \in \mathbb{C}$, when we change the arg z keep changing the arg t to keep integral convergent.

Directions '$t \in \mathbb{C}$ where' Borel transform has singularities, and we can only define lateral Borel transforms 'are called Stokes lines

Start $0 \le \arg z < \pi \Leftrightarrow 0 \le \arg x < \frac{2\pi}{3}$

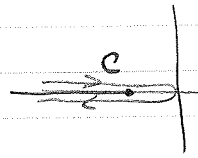


The analytic continuation $S\Phi_1$ is given by $S_{-\theta}\Phi(z)$

But when $\arg z = \pi \Leftrightarrow \arg x = 2\pi/3$: Stokes line

How do we jump it? We need to relate S_+ and S_- (summations before and after the Stokes line)

$$(S_+ - S_-)\Phi_1 = \int_C dt\, e^{-zt} B[\Phi_1](t)$$



But as we mentioned $B[\Phi_1](t)$ has a Log-branch cut along negative real axis, starting at $t=-A$.

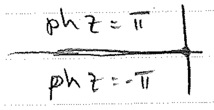
In fact it is not difficult to see that

$$B[\check{\Phi}_1](t)\Big|_{t=-A} = \frac{-i}{2\pi i}\left(\frac{1}{t+A} + \underbrace{\psi(t+A)}_{\substack{\text{Regular at}\\ t=-A}} \log(t+A) + \dots\right)$$

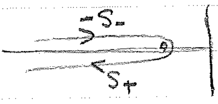
$\Leftrightarrow \psi(t)$ regular at $t=0$

Thus $$(S_+ - S_-)\Phi_1 = \oint_{-A} dt\, e^{-zt}\left(\frac{-i}{2\pi i}\frac{1}{t+A}\right) + \frac{-i}{2\pi i}\int_C dt\, e^{-zt}\psi(t+A)\log(t+A)$$

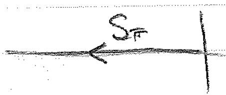
<u>Note</u>: • along the negative real axis $\ln z = \ln|z| + i\,\mathrm{ph}z$



$$\Rightarrow \int_C dt\, f(t+A)\log(t+A) = \int_C dt\, f(t+A)\underset{\to 0}{\log|t+A|} + \int_{S_+} dt\, f(t+A)(-i\pi) - \int_{S_-} dt\, f(t+A)(i\pi)$$



$$= -2\pi i\int_0^{-\infty} dt\, f(t) \mathbin{/\!/}$$



$$\Rightarrow \int_C dt\, e^{-zt}\psi(t+A)\log(t+A) = (-2\pi i)\, e^{+zA}\int_0^{-\infty} dt\, e^{-zt}\psi(t) \mathbin{/\!/}$$

• $$\int_C \frac{dt\, e^{-zt}}{t+A} = \oint_{-A}\frac{dt}{t+A}e^{-zt} = -2\pi i\, e^{zA} \mathbin{/\!/}$$

$$S_{\pi+}\check{\Phi}_1 = S_{\pi-}\check{\Phi}_1 + \textcircled{i}\, e^{zA}\left(1 + \int_0^{-\infty} dt\, e^{-zt}\psi(t)\right)$$

The analytic continuation across the Stokes line of Φ_1 includes Φ_1 <u>plus</u> an extra exponential!

CROSSING A STOKES LINE TURNS ON AN EXP TERM, HIGHLY SUPPRESSED: STOKES PHENOMENA

APPEARANCE OF STOKES CONSTANT

Then... What is $\psi(t)$? Let's take a closer look at the behaviour of Borel transforms:

$$\psi(t+A) = \frac{5}{48} + \frac{385}{4608}(t+A) + \frac{85085}{132704}(t+A)^2 + a_3(t+A)^3 + \dots$$

$$B[\Phi_2](t)\Big|_{t=0} = \underline{\frac{5}{48}} + \underline{\frac{385}{4608}}\,t + \underline{\frac{85085}{132704}}\,t^2 + \underline{a_3}\,t^3 + \dots$$

Thus $\Psi(t) = B[\Phi_2](t)$!

$$B[\check{\Phi}_1](t)\Big|_{\underbrace{t=-A}} \simeq -\frac{\boxed{S_1}}{2\pi i}\left[\frac{1}{t+A} + \underbrace{B[\check{\Phi}_2](t+A)\log(t+A)}\right] + \dots$$

at the singularity Stokes constant second saddle at distance e^{zA}

Borel transform expanded at the singular point $t=A$ encodes the perturbative expansion of the "saddle"/solution appearing in the transseries at exponential distance e^{zA}, in the form of its Borel transform multiplying the log cut singularity!

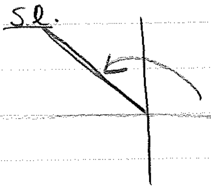
$$S_{-\pi^+}\check{\Phi}_1(z) = S_{-\pi}\check{\Phi}_1(z) + i e^{zA}\left(1 + \int_0^{-\infty} dt e^{-zt} B[\check{\Phi}_2](t)\right)$$

$$= S_{-\pi^-}\check{\Phi}_1(z) + i e^{zA}\left(\textcircled{1} + S_{\pi}\check{\Phi}_2\right)$$

NOTE $S_\theta \Phi = c_0 + S_\theta \check{\Phi}$ (as we had to remove c_0)

$\Rightarrow S_{-\pi^+}\Phi_1 = S_{-\pi^-}\Phi_1 + i e^{zA} S_{-\pi}\Phi_2$ evaluated at $z = |z|e^{i\pi}$.

CONSEQUENCE : Transseries $y(x) = c_1 y_1 + c_2 y_2$



Take $\sigma_2 = 0$ $\sigma_1 = 1$ at $\arg x = 0$

- From $0 < \arg x < 2\pi/3$ solution will be given by a Borel summation of y_1

$$S y(x) = \sigma_1 \frac{e^{-A/2\, x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} S\Phi_1(x) \qquad \sigma_1 = 1$$

- Crossing the stokes line $\arg x = 2\pi/3$

$$S_+\Phi_1 = S_-\Phi_1 + i e^{x^{3/2} A} S\Phi_2$$

$$S_+ y(x) = \frac{e^{-A/2\, x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} S_-\Phi_1 + \frac{i e^{A/2\, x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} S\Phi_2$$

$$= S_- y_1 + i S y_2 \qquad \leftarrow \quad \sigma_2 \text{ jumped from } \underline{0 \to i}$$

appearance of two exponentials

- We can now take $2\pi/3 < \arg x < \pi$ ← negative real axis

$$S_\pi y = S_\pi y_1 + i S_\pi y_2 = \underbrace{\frac{e^{-A/2\, x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}} S\Phi_1 + i \underbrace{\frac{e^{A/2\, x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}} S\Phi_2$$

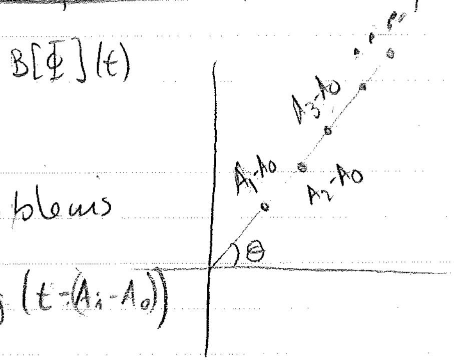
Taking the leading behaviour of Φ_1 and Φ_2 for $|x| \to \infty$ $\arg x = \pi$

$$\Rightarrow S_\pi y \simeq \frac{e^{+A/2 |x|^{3/2} i}}{2\sqrt{\pi}\,|x|^{1/4}} e^{-i\pi/4} + i \frac{e^{-A/2|x|^{3/2} i}}{2\sqrt{\pi}\,|x|^{1/4}} e^{-i\pi/4} + \cdots$$

$$\simeq \frac{\sin(A/2 |x|^{3/2} + \pi/4)}{\sqrt{\pi}\,|x|^{1/4}}$$ as expected!

This is a General feature of Borel transforms along Stokes lines

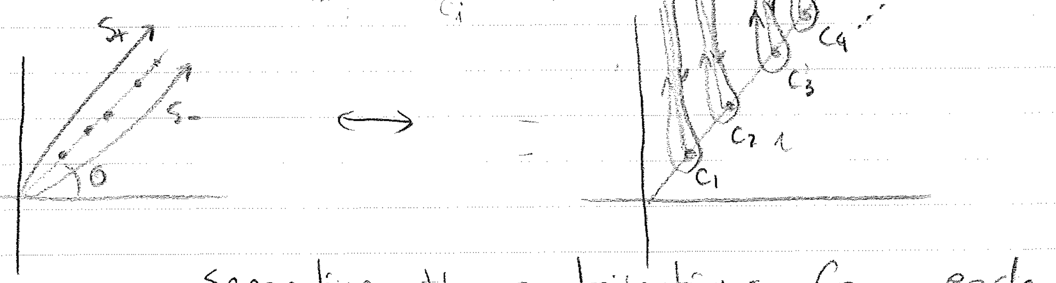
Transseries with more than 1 singularity along a Stokes line is common in non-linear problems



- $B[\Phi_0](t)\Big|_{t=A_i} \simeq + \frac{S_{A_0 \to A_i}}{2\pi i} B[\Phi_{A_i}](t-(A_i - A_0))\log(t-(A_i-A_0))$

- $F \sim \sum e^{-A_i z} \sigma_i \Phi_i(z)$

expected transseries



Separating the contributions from each singular branch cut.

$$(S_{\theta^+} - S_{\theta^-})\Phi_j = -\sum_i S_{A_j \to A_i} \int_{C_i} dt\, e^{-zt} B[\Phi_{A_i}](t - (A_i - A_j))$$

$$= -\sum_i S_{A_j \to A_i} e^{-z(A_i - A_j)} S_{\theta^-}\Phi_i$$

$$\Rightarrow (S_{\theta^+} - S_{\theta^-}) F = \sum_j e^{-A_j z} \sigma_j (S_{\theta^+} - S_{\theta^-})\Phi_j$$

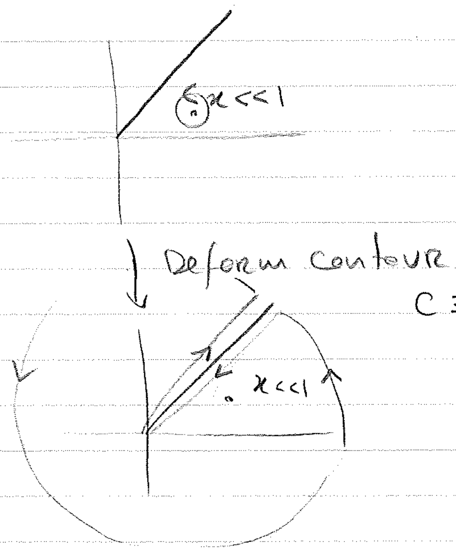
$$= -\sum_{i,j} e^{-zA_i} S_{A_j \to A_i} \sigma_j S_{\theta^-}\Phi_i \quad //$$

Definition $\boxed{S_{\theta+} - S_{\theta-} = - S_{\theta-} \circ \mathrm{Disc}_\theta}$

$\boxed{1.3.3}$ Large order relations

Given a function $\Phi(x)$ with a discontinuity along some ray starting from origin on complex plane in some direction θ, Cauchy's theorem translates to

$$\Phi(x) = -\frac{1}{2\pi i}\int_0^{e^{i\theta}\infty} dw\, \frac{\mathrm{Disc}_\theta \Phi(w)}{w-x}$$



$$\Phi(x) = +\frac{1}{2\pi i}\oint_x \frac{\Phi(w)}{w-x}dw$$

$$= +\underbrace{\frac{1}{2\pi i}\oint_\infty dw\,\frac{\Phi(w)}{w-x}}_{\text{vanishes in general}} + \underbrace{\frac{1}{2\pi i}\int_C dw\,\frac{\Phi(w)}{w-x}}_{S_+ - S_- = -S_-\circ \mathrm{Disc}_\theta}$$

$C \equiv S_+ - S_-$

$$= -\frac{1}{2\pi i}\int_0^{e^{i\theta_-}} dw\,\frac{\mathrm{Disc}_\theta \Phi(w)}{w-x}$$

- $\Phi(x)$ is an expansion in small x
- $\mathrm{Disc}\,\Phi(w)$ is also expansion in small w
- $\frac{1}{w-x}$ can be expanded for small x

<u>Airy function</u>: $\Phi_1(y) = \sum \frac{(-1)^n C_n}{(A/2)^n} y^n \qquad y = x^{-3/2}$

$$(S_+ - S_-)\,\Phi_1 = i e^{A/y} S\,\Phi_2 = -S\circ \mathrm{Disc}\,\Phi_1$$

arg $x = \frac{2\pi}{3}$
arg $y = -\pi$

$$\Rightarrow \mathrm{Disc}_\pi \Phi_1(y) = -i e^{A/y}\Phi_2 = -i e^{A/y}\sum \frac{C_n}{(A/2)^n} y^n$$

Applying Cauchy's thm:

$$\Phi_1(y) = +\frac{i}{2\pi i}\int_0^{e^{-i\pi}\infty} dw\, e^{A/w}\frac{\Phi_2(w)}{w-y} \qquad d\bar{w}$$

$$\parallel$$

$$\sum\frac{(-1)^n C_n y^n}{(A/2)^n} = \frac{i}{2\pi i}\int_0^{e^{-i\pi}\infty} dw\, e^{A/w}\sum_{n\geq 0}\frac{y^n}{w^{n+1}}\sum_{k\geq 0}\frac{C_k w^k}{(A/2)^k}$$

$$= \frac{i}{2\pi i}\sum_{n\geq 0} y^n \sum_{k\geq 0}\frac{\Gamma(n-k)}{(-A)^{n-k}}\frac{C_k}{(A/2)^k} \quad (\text{only valid } n\geq k)$$

Then at large order (assume n large so $n > k$)

$$\underbrace{\frac{(-1)^n c_n}{(A/2)^n}}_{\substack{a_n^{(1)} \text{ coefficient } (n \gg 1) \\ \text{of } \Phi_1}} \simeq \frac{i}{2\pi i} \sum_{k \geq 0} \frac{\Gamma(n-k)}{(-A)^{n-k}} \underbrace{\frac{c_k}{(A/2)^k}}_{\substack{\text{coeff of } \Phi_2 \quad a_n^{(2)} \\ \text{low order in } k}}$$

$$a_n^{(1)} \simeq \frac{i}{2\pi i} \frac{\Gamma(n)}{(-A)^n} \sum_{k \geq 0} \frac{\Gamma(n-k)}{\Gamma(n)} (-A)^k a_k^{(2)}$$

$$\simeq \frac{i}{2\pi i} \frac{\Gamma(n)}{(-4/3)^n} \left\{ \overset{1}{a_0^{(2)}} - \frac{1}{n-1} A \underbrace{a_1^{(2)}}_{5/48} + \frac{A^2}{(n-1)(n-2)} \underbrace{a_2^{(2)}}_{\frac{385}{4608}} + \cdots \right\}$$

Thus we derived from the Borel transform the Large order relations we saw previously.

SUMMARY

Perturbation theory + Borel transforms

↓ Analyse singularity structure + large order behaviour

Predict non-pert saddles one needs to include for a complete Transseries

↓ Use transseries structure + Discontinuities

Determine Stokes Phenomena and obtain solution in all complex plane.

↓ Summation techniques: (opt truncation, Borel, ...)

Analyse solution globally

$\boxed{1.3.4}$ Saddle point approximation, complex saddles & transseries

I've mentioned that the two solutions of Airy eq are in fact two separate saddles of an integral.

A solution to the Airy equation can be written as

$$y_\gamma(x) = \frac{1}{2\pi i}\int_\gamma du\, e^{-V(u)} \qquad V(u) = -xu + \frac{u^3}{3}$$

γ contour chosen such that integral converges.
2 homologically indep. contours ⇒ 2 indep solutions

Take integral solution $y_\gamma(x)$ as zero-dim path integral and perform <u>saddle-point analysis</u>

▷ Saddle points of $V(u)$: $V'(u) = -x + u^2 = 0 \qquad u_\pm^* = \pm\sqrt{x}$

and $V(u_\pm^*) = \mp \frac{2}{3} x^{3/2}$

Expand integral close to each saddle

$$V(u) = \underbrace{V(u_\pm^*)}_{\mp \frac{2}{3}x^{3/2}} + \frac{1}{2}\underbrace{V''(u_\pm^*)}_{\pm 2\sqrt{x}}(u-u_\pm^*)^2$$

$$y_{\gamma_\pm}(x) = \frac{1}{2\pi i}\, e^{\pm \frac{2}{3}x^{3/2}} \int_\gamma du\, e^{\mp\sqrt{x}(u-u_\pm^*)^2}$$

Which is dominant solution? Depends on arg x

▷ Steepest descent contour for each saddle u^*

γ: passes through u^*
$\mathrm{Im}(V(u) - V(u^*)) = 0 \quad , u \in \gamma$
$\mathrm{Re}(V(u)) \to +\infty$ when $u \to \infty$

Plot these contours normalized to $V(u^*)$

| Brown $\mathrm{Re}(V(u) - V(u^*)) > 0$
| Blue < 0
} γ needs to finish in brown area

Vary $\arg x = \theta$

| dashed subleading saddle
| Solid line leading saddle
red circle saddle from which we are normalizing

Blue saddle is contributing zero
Purple saddle is "turned on".

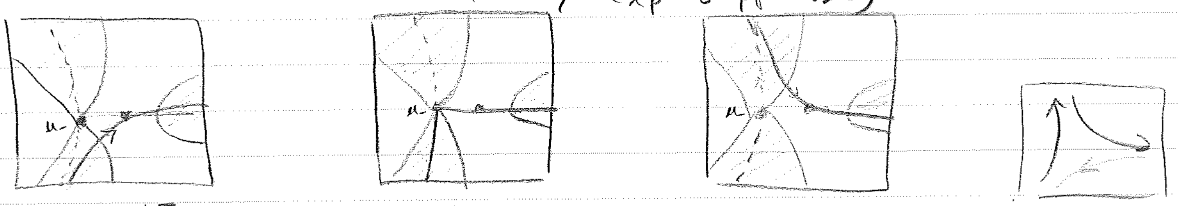
What we find:

→ General steepest descent contour : single contour passes through each saddle

→ When $\mathrm{Im}\,(V(\mu_+^*) - V(\mu_-^*)) = 0 \Rightarrow \mathrm{Im}\, x^{3/2} = 0$

$\arg x = l\,\frac{2\pi}{3} \quad l \in \mathbb{Z}$

<u>Stokes line</u> : contour passes through both saddles
the subleading saddle gets picked up!
(very exp suppressed)



$\theta = 0^-$	$\theta = 0$	$\theta = 0^+$
γ_-	$\longrightarrow$	γ_-
γ_+	$\longrightarrow$	$i\gamma_- + \gamma_+$

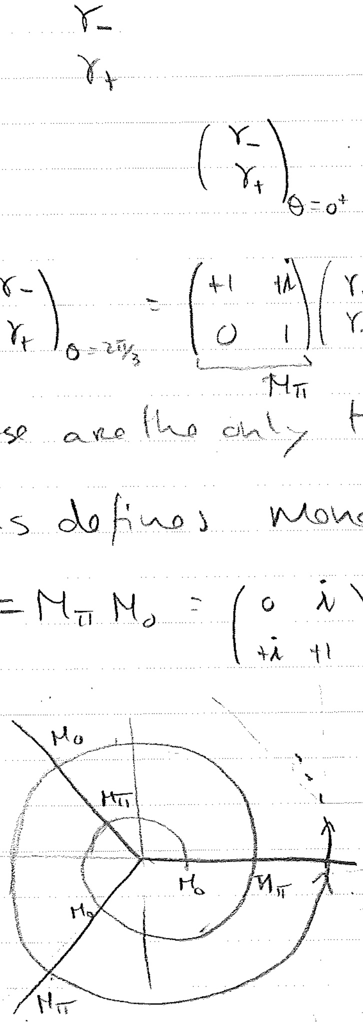
$$\begin{pmatrix}\gamma_- \\ \gamma_+\end{pmatrix}_{\theta=0^+} = \underbrace{\begin{pmatrix} 1 & 0 \\ +i & +1 \end{pmatrix}}_{M_0} \begin{pmatrix}\gamma_- \\ \gamma_+\end{pmatrix} \qquad \arg(x^{3/2}) = 2\pi i + 0/ \quad \left(\arg x = \frac{4\pi i}{3} n\right)$$

$$\begin{pmatrix}\gamma_- \\ \gamma_+\end{pmatrix}_{\theta=2\pi/3} = \underbrace{\begin{pmatrix} +1 & +i \\ 0 & 1 \end{pmatrix}}_{M_\pi} \begin{pmatrix}\gamma_- \\ \gamma_+\end{pmatrix} \qquad \arg x^{3/2} = \pi i + 2\pi i n \quad \left(\arg x = \frac{2\pi}{3} + \frac{4\pi i n}{3}\right)$$

These are the only two types of Stokes lines 0, π type

This defines monodromy matrix

$$\mathcal{M} = M_\pi M_0 = \begin{pmatrix} 0 & i \\ +i & +1 \end{pmatrix} \qquad (M_\pi M_0)^6 = \mathbb{1} \leftarrow \text{Monodromy of Problem}$$



One needs to go 4 times around the complex plane or x to reach the initial point again

$l \in \mathbb{Z}$

→ When $\mathrm{Re}\,(V(\mu_+^*) - V(\mu_-^*)) = 0 \Rightarrow \mathrm{Re}\, x^{3/2} = 0 \quad \arg x = \frac{\pi}{3} + \frac{2\pi}{3} l$

The two saddles are of the same magnitude

<u>Anti-Stokes line</u>

- when crossing anti-Stokes lines the saddles change dominance.
- Exactly at anti-Stokes lines oscillatory behaviour

Stokes lines: Mathematically transseries parameters jump, an exp supressed exponential appears
Physically asymptotically nothing happens

anti-Stokes lines: Mathematically nothing happens
Physically assymptotics completely different (exponentials imaginary)
→ Phase transitions / Lee-Yang zeros --

Can we see the Monodromy from Borel transforms?
Yes!

Stokes lines are of type
- $\arg x^{3/2} = \arg z = 0 + 2\pi l$
- $= \pi + 2\pi l$

At $\arg x^{3/2} = \pi$ $\left(\arg x = \frac{2\pi}{3}\right)$ we saw

$$\left(S_{\pi+} - S_{\pi-}\right)\Phi_1 = i e^{zA} S \Phi_2 \Rightarrow \mathrm{Disc}_\pi \Phi_1 = -i e^{zA} \Phi_2$$

$$\left(S_{\pi+} - S_{\pi-}\right)\Phi_2 = 0 \Rightarrow \mathrm{Disc}\,\Phi_2 = 0$$

Exercise Show
- $\left(S_{0+} - S_{0-}\right)\Phi_2 = i e^{-zA} S_0 \Phi_1 \Rightarrow \mathrm{Disc}_0 \Phi_2 = -i e^{-zA}\Phi_1$
- $\left(S_{0+} - S_{0-}\right)\Phi_1 = 0$

Take the transseries $y(x,\sigma_1,\sigma_2) = \sigma_1 y_1(x) + \sigma_2 y_2(x)$

$$S_{0+} y(x) = \sigma_1 S_{0+} y_1 + \sigma_2 S_{0+} y_2 = \sigma_1 S_{0-} y_1 + \sigma_2 \left(S_{0-} y_2 + i S_{0-} y_1\right)$$

$$= (\sigma_1 + i\sigma_2) S_{0-} y_1 + \sigma_2 S_{0-} y_2 = S_{0-} y(x, \sigma_1 + i\sigma_2, \sigma_2)$$

$$S_{\pi+} y(x) = \sigma_1 S_{\pi+} y_1 + \sigma_2 S_{\pi+} y_2 = \sigma_1 S_{\pi-} y_1 + (\sigma_2 + i\sigma_1) S_{\pi-} y_2$$

$$= S_{\pi-} y(x, \sigma_1, \sigma_2 + i\sigma_1)$$

Defining operation $S_{\theta+} = S_{\theta-}\circ(\mathbb{1} - \mathrm{Disc}_\theta) \equiv S_{\theta-}\circ\sigma_\theta$

Stokes Automorphism

$$\underline{\sigma_0} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 + i\sigma_2 \\ \sigma_2 \end{pmatrix} \Rightarrow \underline{\sigma_0} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

$$\sigma_\pi \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 + i\sigma_1 \end{pmatrix} \Rightarrow \underline{\sigma_\pi} = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$$

$$\sigma_\pi \sigma_0 = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix} = m \qquad m^6 = \mathbb{1} /\!/$$

We obtain the same monodromy.

$\boxed{2}$ ECALLE'S RESURGENCE

$\boxed{2.1}$ Summarize previous results but generalising to non-linear problems

Let us look at a problem inspired by relativistic hydrodynamics describing strongly coupled systems

→ Strongly coupled fluid such as quark-gluon plasma in particle accelerators. After the initial (far-from-eq) period the system begins to thermalize and rel. hydro can be used as an effective theory describing the transition to equilibrium at late times.

Assuming enough symmetry: expanding plasma, conformal, with longitudinal boost invariance the equations of conservation of energy momentum tensor simplify and one can get Energy density / pressure / Effective temperature as a late time expansion where the terms of the expansion are given by the transport coefficients introduced in the shear stress tensor (dissipative effects)

The transport coefficients are determined by the microscopic theory behind the plasma. If this is SYM strongly coupled then one can use holography and calculate the late time series as a perturbation around a black hole solution on gravity side!

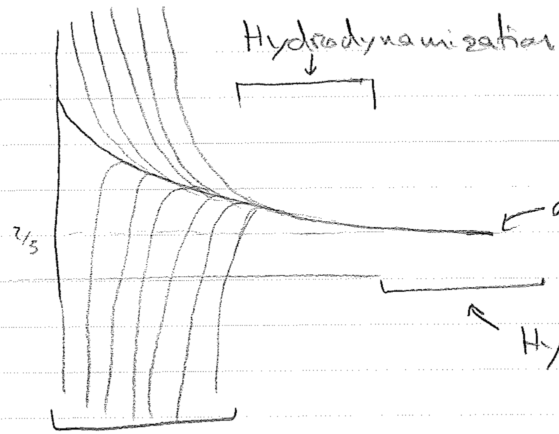
Toy model for these systems solve $\nabla_\mu T^{\mu\nu} = 0$ with above symmetries and upgrade shear stress tensor to dynamical field to obtain corrections to ideal fluid Hydrodynamic gradient expansion (Muller-Israel-Stewart approach):

Non-linear 1st order ODE describing pressure anisotropy $z \sim \tau$ proper time $f \sim P_L - P_R$ pressure anis.

$$z\, C_{\tau\pi} f f' + 4 C_{\tau\pi} f^2 + \left(z - \frac{16 C_{\tau\pi}}{3}\right) f - 4\frac{C_\eta}{9} + \frac{16 C_{\tau\pi}}{9} - \frac{2z}{3} = 0$$

$\underset{\uparrow}{C_{\tau\pi}}$, C_η ↖ transport coeff

relaxation time of shear stress tensor



Hydrodynamization time : convergence to hydro description although system still far-from equilibrium

attractor

Hydro description independent of initial conditions

Early time dictated by initial conditions

Example of transseries solution at late times

early times : exponentials all of same order
Late times : evolution given by divergent perturbative series of viscous hydro
Hydrodynamizatio: system very anisotropic but all solutions exponentially close to each other

---

Let us study this NLODE : $(z \sim \tau \gg 1$ proper time$)$

Assume an ansatz of form $e^{-Az}\Phi(z)$ $\Phi(z)=z^{\beta}\sum c_n z^{-n}$

$\Rightarrow A = \frac{2}{3C_{\tau\pi}}$, $\beta = \frac{C_\eta}{C_{\tau\Pi}}$

The most general solution is not a single exponential, but an infinity of them

$$\mathcal{F}(z,\sigma) = \sum \left(\sigma e^{-Az} z^{\beta}\right)^n \Phi^{(n)}(z) \qquad a_0^{(0)} = 2/3$$

$$\Phi^{(n)}(z) = \sum_{k\geq 0} a_k^{(n)} z^{-k}$$

↑

Transseries , σ - transseries parameter determined by initial conditions / boundary conditions

Plugging this inte ODE $\Rightarrow$ recursion relation for $c_k^{(n)}$

This solution does give the expected attractor solution

- At $z\to\infty$ $\mathcal{F}(z,\sigma) \sim c_0^{(0)} = 2/3$

- At early times exponentials become of $\mathcal{O}(1)$ and govern the behaviour.

Let us analyse the perturbative expansion

Exercise The coefficients can be obtained from

$$a_0^{(0)} = 2/3 \qquad a_1^{(0)} = 4\frac{C_\eta}{9}$$

$$a_{k+1}^{(0)} = C_{\tau\pi}\left(\frac{16}{3}a_k^{(0)} - \sum_{n=0}^{k}(4-n)\,a_{k-n}^{(0)}a_n^{(0)}\right) \qquad k\geq 1$$

NOTE: The microscopic theory describing the fluid determines phenomenological parameters.
If it is SYM we have

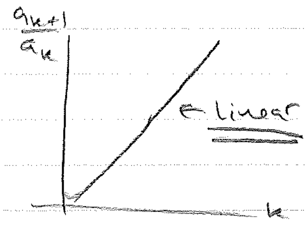
$$C_{\tau\pi} = \frac{2-\log 2}{2\pi} \qquad C_\eta = \frac{1}{4\pi}$$

We can generate several $a_k^{(0)}$, and we know that they should have a universal behaviour

$$a_k^{(0)} \sim \frac{\Gamma(k-\beta)}{A^{k-\beta}}\frac{S_1}{2\pi i}\left(1+\dots\right)$$

How can we see this? Can we determine β, A, S_1?

$\boxed{2.1.1}$ Plot $\frac{a_{k+1}^{(0)}}{a_k^{(0)}}$ to see the factorial growth.



Knowing the universal behaviour

$$a_k^{(0)} \sim \frac{\Gamma(k+\beta)}{A^{k+\beta}}\frac{S_1}{2\pi i}\left(1+\frac{d_1}{k}+\frac{d_2}{k^2}+\dots\right)$$

$$\frac{a_k}{a_{k+1}} \sim \frac{A}{k+\beta}\,\frac{\sum_{\ell\geq 0} d_\ell/k^\ell}{\sum_{\ell\geq 0} d_\ell/(k+1)^\ell} \underset{k\gg 1}{\sim} \frac{A}{k} - \frac{A\beta}{k^2} + \frac{A(\beta^2+d_1)}{k^3} + \frac{f(d_1,d_2)}{k^4} + \dots$$

Then $\frac{a_k}{a_{k+1}}k \sim A + \mathcal{O}(1/k)$

Use extrapolation methods such as Richardson transforms

NOTE : Richardson transforms assume sequence

$$S(n) \simeq s_0 + \frac{s_1}{n} + \frac{s_2}{n^2} + \dots$$

N-Richardson transform converging to s_0 is given by recursion:

$$RT_S(n,0) = S(n)$$
$$RT_S(n,N) = RT_S(n+1, N-1) + \frac{n}{N}\left(RT_S(n+1,N-1) - RT_S(n,N-1)\right) \qquad N \geq 1$$

This efectively cancels subleading terms in $S(n)$ up to order n^{-N-1}, accelorating convergence.

Example N=1

$$RT_S(n,1) = S(n+1) + n\left(S(n+1) - S(n)\right) =$$
$$= S_0 + \frac{S_1}{n+1} + \frac{S_2}{(n+1)^2} + \dots + n\left(S_0 + \frac{S_1}{n+1} + \frac{S_2}{(n+1)^2} + \dots - S_0 - \frac{S_1}{n} - \frac{S_1}{n^2} + \dots\right)$$
$$\underset{\text{langen}}{=} S_0 - \frac{S_2}{n^2} + \dots$$

- We can check the value of β as well. Define a new sequence

$$\left(\frac{a_k k}{A a_{k+1}} - 1\right) k \sim \underline{\underline{-\beta}} + \frac{(\beta^2 + d_1)}{k}$$

and determine β with extrapolation methods

- To find the subleading coefficients of large order behaviour we just need to iterate this process

$d_1, d_2, \dots$ (which we will see are related to the $e^{-Az}\Phi^{(1)}$ sector in the transseries)

NOTE we can find the coefficients of the exponential sectors from the NLODE.
Recursion relation (Linear)

$$C_{\tau\pi} \sum_{m=0}^{\ell} \left(\Phi^{(m)\prime} - mA\Phi_1^{(m)}\right)\Phi^{(\ell-m)} + 4C_{\tau\pi}\sum_{m=0}^{\ell}\Phi^{(m)}\Phi^{(\ell-m)}$$
$$+ \left(z - \frac{16C_{\tau\pi}}{3}\right)\Phi^{(\ell)} = 0$$

$$\underline{\ell=1} \quad \left(\Phi^{(1)\prime} - A\Phi^{(1)}\right)\Phi^{(0)} + \Phi^{(1)}\Phi_0^{(0)\prime} + 8\,\Phi^{(0)}\Phi^{(1)} + \left(\frac{z}{C_{\tau\pi}} - \frac{16}{3}\right)\Phi^{(1)} = 0$$

- Can we also find the constant $\frac{S_1}{2\pi i}$ of large order behaviour? Yes once we know A, β.

$$\frac{a_k A^{n+\beta}}{\Gamma(n+\beta)} \sim \frac{S_1}{2\pi i} + O(1/n) \qquad S_1 \sim 5.4703\,i$$

2.1.2. Borel transform & singularities

We want to determine $B[\Phi_0](t)$

But given that we know $a_n^{(0)} \sim \Gamma(n+\beta)$

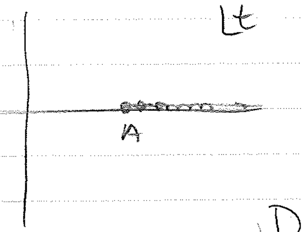
$$B[\Phi_0 z^{-\beta}](t) = \sum a_n^{(0)} \frac{t^{n+\beta-1}}{\Gamma(n+\beta)}$$

Unlike the previous case, we cannot determine the Borel transform in closed form. But we can approximate it, using a finite number of terms and an analytic approximation such as Padé approximants

▶ $BP_N[\Phi](t) = \dfrac{\sum_{\ell=0}^{N/2} a_n t^n}{1+\sum_{\ell=1}^{N/2} b_n t^n}$ where a_n, b_n match the expansion Φ to N terms

We can plot this padé approximant, in particular its poles, and these will give us singular directions, or Stokes directions, in the complex Borel plane.

An accumulation of poles indicates the existence of branch cut singularity.



Branch cut starting at $A = t$.
In the positive real axis.

Do we have extra branch cuts?
Difficult to see in this case. Can we transform the above function to make the extra branch cuts visible (they are all on top of each other)

Perform a conformal transformation

▶ Once we have the Borel transform change variables:

$$t = -\frac{2Az}{(1-z)^2}$$

Map singularities

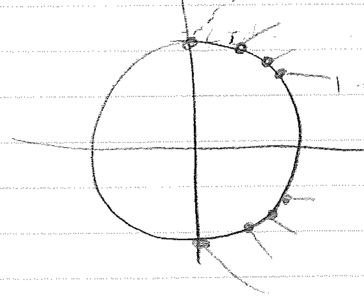
$t=0 \longrightarrow z=0$

$t=A \longrightarrow z=\pm i$

$t=2A \longrightarrow z=\frac{1}{2}(1\pm i\sqrt{3})$

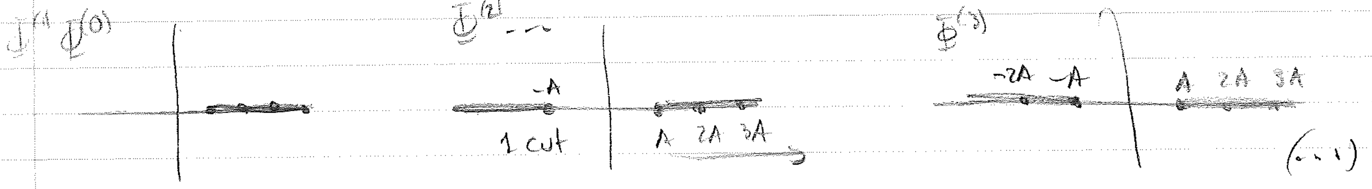
$t=3A \longrightarrow z=\frac{1}{3}(2\pm i\sqrt{5})$

$t=4A \longrightarrow z=\frac{1}{4}(3\pm i\sqrt{7})$ ...



Expand around $z=0$ and calculate Padé approximant
$\Rightarrow$ appearance of multiple sing.

In general if we were to study the asymptotics of the different sectors $\Phi^{(m)}(z)$ associated to the respective exponential weight e^{-mAz}, we would find



This means that when one studies the Borel transforms associated to each of the pert/non-pert sectors we have

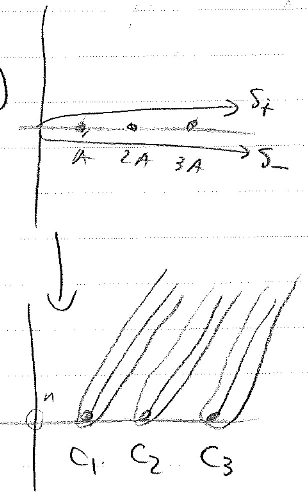
$$\boxed{\begin{aligned} \mathcal{B}[\Phi^{(n)}](t)\Big|_{t=kA} &\simeq \frac{S_{n\to n+k}}{2\pi i}\, \mathcal{B}[\Phi^{(n+k)}](t-kA)\, \log(t-kA) \\ &\text{with} \quad k>-n \quad (k\neq 0) \end{aligned}}$$

$S_{n\to n+k}$ are functions of the Stokes Constants $\to$ (Borel residues)

These can be read from an analysis of Borel transforms around all singularities, but can also be obtained more systematically through the the framework of Écalle's resurgence

Using this framework we can decrease the degree of the # of unknows coefficients $S_{n\to n+k}$ dramatically as they are quite constrained.

Writing the Discontinuities associated to Stokes phenomena



$$\left(S_{0^+} - S_{0^-}\right)\Phi^{(n)} = -\sum_{k>0} S_{n\to n+k}\int_{C_k} dt\, e^{-zt}\, \mathcal{B}[\Phi^{(n+k)}](t-kA)$$

$$= -\sum_{k>0} S_{n\to n+k}\, e^{-zkA}\, S_{0^-}\Phi^{(n+k)}$$

$$:= -S_{0^-}\,\mathrm{Disc}_0\,\Phi^{(n)}$$

$$\Rightarrow \mathrm{Disc}_0\,\Phi^{(n)} = \sum_{k\geq 0} S_{n\to n+k}\, e^{-zkA}\,\Phi^{(n+k)}$$

NOTE: For $n\geq 2$ there is also a discontinuity in the negative real axis

$$\mathrm{Disc}_\pi\,\Phi^{(n\geq 2)} = \sum_{k=1}^{n-1} S_{n\to n-k}\, e^{zkA}\,\Phi^{(n-k)}$$

$\boxed{2.1.3}$ Écalle's Resurgence

A different way of writing the Discontinuity is

$$\underbrace{\mathbb{1} - \text{Disc}_0 \equiv \mathfrak{S}_0}_{\text{Stokes automorphism}} = \exp\Big\{ \underbrace{\sum_{n\geq 1} e^{-zAn} \Delta_{An}}_{\text{Derivation}} \Big\}$$

Δ_{nA} is called "alien derivative"

(Think of usual automorphism that generates translations $\exp(\mathrm{ad}_x)$)

→ nA are all possible singularities in $\mathbb{R}^+$ of Borel plane

$$\Rightarrow \sum_{k\geq 1} S_{n\to n+k}\, e^{-zkA}\, \Phi^{(n+k)} = \Big(\mathbb{1} - \exp\Big\{\sum_{k\geq 1} e^{-zkA} \Delta_{kA}\Big\}\Big)\Phi^{(n)}$$

$$= -\sum_{k\geq 1} \frac{1}{k!}\Big(\sum_{n\geq 1} e^{-znA}\Delta_{nA}\Big)^k \Phi^{(n)}$$

Match exp weights :

$$e^{-zA} : \quad \Delta_A \Phi^{(n)} = -S_{n\to n+1}\, \Phi^{(n+1)}$$

$$e^{-2Az} : \quad \Delta_{2A}\Phi^{(n)} + \frac{1}{2}\Delta_A^2 \Phi^{(n)} = -S_{n\to n+2}\, \Phi^{(n+2)}$$

$$\vdots \qquad \Leftrightarrow \Delta_{2A}\Phi^{(n)} = -S_{n\to n+2}\,\Phi^{(n+2)} - \frac{1}{2} S_{n\to n+1} S_{n+1\to n+2}\, \Phi^{(n+2)}$$

The Δ_{nA} are in some way retrieving the algebraic information of the Borel transforms, removing their functional nature.

But it can be proven that

$$\Delta_{kA}\Phi^{(n)} = S_k\,(n+k)\,\Phi^{(n+k)} \qquad k \leq 1 \quad k \neq 0$$

Where $\{S_k\} = \{S_1, S_{-1}, S_{-2}, \ldots\}$ are the minimal set of unknown <u>Stokes constants</u> which are needed to retrieve Stokes phenomena:

These "resurgence relations" follow directly from the so-called "bridge equations", which are usually the starting point of any discussions of resurgence (they are called bridge equations because they bridge alien calculus with usual one.

A very crucial property is that those relations truncate at $k=1$, unlike the Borel residues $S_{n\to n+k}$.

We can then find

$$S_{n\to n+1} = -(n+1)S_1$$

$$S_{n\to n+2} = -\frac{1}{2}(n+1)(n+2)S_1^2$$

$$\vdots$$

$$\boxed{S_{n\to n+k} = -\frac{1}{k!}\frac{(n+k)!}{n!}S_1^k}$$

dependent on a single Stokes constant!

$$\Rightarrow \mathrm{Disc}_0\,\Phi^{(n)} = -\sum_{k\geq 1}\frac{1}{k!}\frac{(n+k)!}{n!}S_1^k\, e^{-kAz}\,\Phi^{(n+k)}$$

<u>Exercise</u> Show that for the transseries

$$S_{0+}F(z,\sigma) = S_{0-}(\mathbb{1}-\mathrm{Disc}_0)F(z,\sigma) = \sum_{n\geq 0} e^{-nAz}(\sigma+S_1)^n S_{0-}\Phi^{(n)}$$

$$= S_{0-}F(z,\underbrace{\sigma+S_1}_{\text{jump of }\sigma.})$$

On the negative real axis there will be also a Stokes line for $\Phi^{(n\geq 2)}$ thus we will also have

$$(S_{\pi+}-S_{\pi-})\,\Phi^{(n\geq 2)} = -S_{\pi-}\,\mathrm{Disc}_\pi\,\Phi^{(n\geq 2)}$$

This will depend on all the Stokes constants $S_{-1},\dots,S_{-(n-1)}$ //

2.1.4 Large order relations

Once we know the $\mathrm{Disc}_0\,\Phi^{(n)}$, $\mathrm{Disc}_\pi\,\Phi^{(n)}$ we can apply the Cauchy theorem

$$\Phi^{(n)}(x=1/z) = -\frac{1}{2\pi i}\int_0^{\infty}dw\,\frac{\mathrm{Disc}_0\,\Phi^{(n)}(w)}{w-x} - \frac{1}{2\pi i}\int_0^{-\infty}dw\,\frac{\mathrm{Disc}_\pi\,\Phi^{(n)}(w)}{w-x}$$

- Use small x expansion for $\Phi^{(k)}(x)$
- $\frac{1}{w-x} = \frac{1}{w}\sum_{k\geq 0}\left(\frac{x}{w}\right)^k \quad x \ll 1$
- Integrate and compare equal powers of x

14 − 21 − 27
②

<u>NOTE</u>: $d_z S\Phi = S(d_z\Phi)$

- $d_z S\Phi = \int_0^{e^{i\theta}\infty} ds\, B[\Phi](s)(-s)e^{-sz} = -\int ds\, B[\partial_z\Phi](s) = S(\partial_z\Phi)$

- $S_+ - S_- = -S_-\,\mathrm{Disc}$

  $S_+ = S_-\,\underline{\sigma}_\theta$

$\sigma_\theta = e^{\underline{\Delta}_\theta}$ $\quad$ $\underline{\Delta}_\theta$ directional differentiation, decomposed into components depending only on singularities in complex plane along direction θ

$$\sigma_\theta = \exp\Big\{ \sum_{\omega_i \in \mathrm{Sing}_\theta} \underbrace{e^{-\omega_i z}\,\Delta_{\omega_i}}_{\text{differentiation (obeys Leibnitz rule)}} \Big\}$$

<u>NOTE 2</u> Several properties of Borel transforms

Denote $\mathbb{C}[[z^{-1}]]$ set of all power series in z^{-1}

$$B : z^{-1}\mathbb{C}[[z^{-1}]] \longrightarrow \mathbb{C}[[s]]$$

$$\Phi = \sum_{n\geq 0} a_n z^{-n-1} \longrightarrow B[\Phi](s) = \sum_{n\geq 0} a_n \frac{s^n}{\Gamma(n+1)}$$

multiplicative model	convolutive model
$\Phi \longrightarrow$	$B[\Phi]$
$z^{-\alpha-1}$	$s^\alpha / \Gamma(\alpha+1)$
$\partial_z \Phi$	$-s\,B[\Phi]$
$z\,\Phi$	$\partial_s B[\Phi]$
$\Phi_1\,\Phi_2$	$B[\Phi_1] * B[\Phi_2]$ $= \int_0^s d\tilde{s}\, B[\Phi_1](\tilde{s})\, B[\Phi_2](s-\tilde{s})$
$\Phi(\lambda z)$	$\lambda\, B[\Phi](\lambda^{-1}s)$
$\Phi(z+\lambda)$	$e^{-\lambda s}\, B[\Phi](s)$

11-21-22
②

A property of $\dot{\Delta}_\omega = e^{-\omega z} \Delta_\omega$ is

$[\dot{\Delta}_\omega, \partial/\partial z] = 0$
$[\partial_z, \partial_\sigma] = 0$

$\Rightarrow \boxed{\dot{\Delta}_\omega F = S_\omega(\sigma) \frac{\partial F}{\partial \sigma}}$ Bridge equs

because they obey same linearised ODE w.r.t $\partial/\partial z$.

$$S_\omega(\sigma) = \sum_{k\geq 0}^{\infty} S_\omega^{(k)} \sigma^k$$

For $\Phi^{(0)}$: $x \equiv 1/z$

$$\sum a_n^{(0)} x^n = -\frac{1}{2\pi i}\int_0^\infty d\omega \sum_{n\geq 0}\frac{x^n}{\omega^{n+1}}\left(-\sum_{k\geq 1} S_1^k e^{-kA/\omega}\sum_{\ell\geq 0} a_\ell^{(k)}\omega^{\ell-\beta k}\right)$$

$$\simeq \frac{1}{2\pi i}\sum_{n\geq 0} x^n \sum_{k\geq 1} S_1^k \sum_{\ell\geq 0} a_\ell^{(k)}\int_0^\infty d\omega\, e^{-kA/\omega}\omega^{\ell-n-\beta k-1}$$

$$\simeq \sum_{n\geq 0} x^n \sum_{k\geq 1} S_1^k \sum_{\ell\geq 0} a_\ell^{(k)}\frac{\Gamma(n+k\beta-\ell)}{(kA)^{n+\beta k-\ell}}$$

$$\Rightarrow \quad a_n^{(0)} \underset{n\gg 1}{\sim} \sum_{k\geq 1}\frac{S_1^k}{2\pi i}\frac{\Gamma(n+k\beta)}{(kA)^{n+k\beta}}\sum_{\ell\geq 0} a_\ell^{(k)}\frac{\Gamma(n+k\beta-\ell)}{\Gamma(n+k\beta)}(kA)^\ell$$

$$\sim \frac{S_1}{2\pi i}\frac{\Gamma(n+\beta)}{A^{n+\beta}}\left(a_0^{(1)}+\frac{a_1^{(1)}A}{n+\beta-1}+\frac{a_2^{(1)}A^2}{(n+\beta-1)(n+\beta-2)}+\cdots\right)$$

$$+\frac{S_1^2}{2\pi i}\frac{\Gamma(n+2\beta)}{(2A)^{n+2\beta}}\left(a_0^{(2)}+\frac{2A\,a_1^{(2)}}{n+2\beta-1}+\cdots\right)$$

$$\vdots$$

These can be used to predict expansions around non-pert sectors.

2.2 Ambiguity Cancelation

As in many situations in physics, our current problem takes $z \sim$ proper time which we are interested to be real positive.

Thus we would like to find summed transseries for $\arg z = 0$. However $\arg z = 0$ is a Stokes line which means that the Borel summation is only defined as lateral sums due to singularities for all $\mathcal{B}[\Phi^{(n)}](t)$ along this line.

This gives rise to a non-perturbative ambiguity

$$\underbrace{S_{0^+}\Phi^{(0)} - S_{0^-}\Phi^{(0)}}_{\text{the same for all }\Phi^{(n)}\text{'s}} = -\sum_{k\geq 1} S_1^k \underbrace{e^{-kAz}\, S_0\Phi^{(k)}}_{\substack{\text{all exponentially small,}\\ \text{non perturbative}}}$$

How can we make sense of sum along positive real axis?

In fact one can define our observable in a truly unambiguous way along the positive real axis!

We know that not just $\Phi^{(0)}$ but all sectors of transseries $\Phi^{(n)}$ lead to ambiguity at $\arg z = 0$, as

$$(S_+ - S_-)\,\Phi^{(\ell)} \neq 0 \quad \forall \ell$$

as all sectors $\Phi^{(\ell)}$ have $\mathrm{Disc}_0\,\Phi^{(\ell)} \neq 0$

However the observable is given by a transseries where all the sectors are weighed by a parameter σ:

$$\rightarrow \quad F(z,\sigma) = \sum \sigma^n e^{-nAz}\,\Phi^{(n)}$$

All the coefficients of $\Phi^{(\ell)}$ are real and as such $(S_{0+} - S_{0-})\Phi^{(\ell)}$ will give a purely imaginary exponentially small ambiguity.

For a general σ complex one will also have

$$(S_{0+} - S_{0-})\,F(\omega,\sigma) \neq 0$$

Nevertheless one can see that there is a proper choice of $\sigma = \sigma_0$ such that the difference between S_{0+} and S_{0-} cancels at the <u>level of the transseries</u>

This is known as a particular summation called median summation.

→ How can we <u>see</u> the cancelation of imaginary ambiguity?

Define "

$$S_{0\pm} = \underbrace{\tfrac{1}{2}\left(S_{0+} + S_{0-}\right)}_{\text{retrieve Re part}} \pm \underbrace{\tfrac{1}{2}\left(S_+ - S_-\right)}_{\text{retrieve Im part}} = \mathrm{Re} \pm i\,\mathrm{Im}$$

Start with perturbative sector only.

$$i\,\mathrm{Im}\,\Phi^{(0)} \equiv \frac{1}{2}\left(S_+ - S_-\right)\Phi^{(0)} = \frac{1}{2}\sum_{k\geq 1} S_1^k\, S_{0-}\,\Phi^{(k)} e^{-kAz}$$

$$= \frac{1}{2}\sum_{k\geq 1} S_1^k\, e^{-kAz}\left(\mathrm{Re}\,\Phi^{(k)} - i\,\mathrm{Im}\,\Phi^{(k)}\right)$$

Now we need to use $i\,\mathrm{Im}\,\Phi^{(k)} = \frac{1}{2}\sum_{\ell\geq 1} S_1^\ell \binom{\ell+k}{\ell} e^{-\ell Az} S_{0-}\,\Phi^{(\ell+k)}$

and continue this process. The leading contributions to the imaginary part of $\Phi^{(0)}$ are

$$i\,\mathrm{Im}\,\Phi^{(0)} = +\frac{S_1}{2}\,\mathrm{Re}\,\Phi^{(1)}e^{-Az} - \frac{1}{4}S_1^3\,\mathrm{Re}\,\Phi^{(3)}e^{-3Az} + \frac{1}{2}S_1^5\,\mathrm{Re}\,\Phi^{(5)}e^{5Az} + \cdots$$

This means that we need to add to Φ_0 a contribution of $-\frac{S_1}{2}\Phi^{(1)}e^{-Az}$ to cancel the leading ambiguity

$$S_{0^-}\left[\Phi^{(0)} + \frac{S_1}{2}\Phi^{(1)}e^{-Az}\right] = \underbrace{\mathrm{Re}\,\Phi^{(0)} - i\frac{S_1}{2}\,\mathrm{Im}\,\Phi^{(1)}e^{-Az}}_{\text{Real}}\;\underbrace{- i\,\mathrm{Im}\,\Phi^{(0)} + \frac{S_1}{2}\,\mathrm{Re}\,\Phi^{1}e^{-Az}}_{\text{Imaginary}}$$

NOTE $i\,\mathrm{Im}\,\Phi^{(1)}e^{-Az} = S_1\,\mathrm{Re}\,\Phi^{(2)}e^{-2Az} - S_1^3\,\mathrm{Re}\,\Phi^{(4)}e^{-4Az}$

$$\Rightarrow \underbrace{S_{0^-}\left[\Phi^{(0)} + \frac{S_1}{2}\Phi^{(1)}e^{-Az}\right]}_{H?} = \mathrm{Re}\,\Phi^{(0)} - \frac{S_1^2}{2}\,\mathrm{Re}\,\Phi^{(2)}e^{-2Az} + \frac{1}{4}S_1^3\,\mathrm{Re}\,\Phi^{(3)}e^{-3Az} + \frac{S_1^4}{2}\,\mathrm{Re}\,\Phi^{(4)}e^{-4Az} + \cdots$$

$$S_{0^-}F(z, S_{1/2} = \sigma)$$

In fact it is! $S_{0^-}F\left(z, \frac{S_1}{2}\right)$ is such that we can see all imaginary terms canceling and we are left with a real result, given by

$$F_{\mathbb{R}}(z, S_{1/2}) = \mathrm{Re}\,\Phi^{(0)} + \frac{S_1^2}{4}\,\mathrm{Re}\,\Phi^{(2)}e^{-2Az} + O(e^{-4Az})$$

In fact it is enough to fix $i\,\mathrm{Im}\,\sigma = S_{1/2}$ $\mathrm{Re}\,\sigma\ \forall$!

Thus in order to cancel ambiguities we need all the exponential corrections (full trans series).

Another way to obtain this is via Stokes automorphism

$$S_{med} = S_{0^-}\circ\mathfrak{S}_0^{1/2} \quad \text{where } \mathfrak{S}_0^{1/2} = \exp\left\{\tfrac{1}{2}e^{-Az}\Delta_A\right\}. \;!$$

(Systematic way of obtaining an unambiguous summation).

$\boxed{2.4}$ General expressions for Discontinuities & large order behaviour.

Using Écalle's resurgence we can obtain the Borel residues $S_{n\to n+k}$ in terms of the Stokes constants $\{S_1, S_{-1}, S_{-2}, S_{-3}, \ldots\}$

With Transseries $F(z,\sigma) = \sum_{n\geq 0} \left(e^{-Az}\sigma z^{\beta}\right)^n \Phi^{(n)}$

we define an "alien chain" connecting two nodes $\boxed{\Phi^{(n)}}$

<u>Define</u> • Step S $S(n\to m) =$

$$\begin{tikzcd} \boxed{\Phi^{(n)}} \arrow[r, bend left] & \boxed{\Phi^{(m)}} \end{tikzcd}$$

connecting nodes

- Weight of Step S : basically given by $\Delta_{kA}\Phi^{(n)} = S_k (n+k)\Phi^{(n+k)}$

  $\omega(S(n\to m)) = m S_{m-n}$ (See Figure)

- Path P : trajectory composed by any number of steps, connecting two noded $\Phi^{(m)}$ to $\Phi^{(k)}$

  $$P = S_1 \cup S_2 \cup \ldots \cup S_\ell = \begin{tikzcd} \boxed{\Phi^{(m)}} \arrow[r, bend left] & \cdots \arrow[r, bend left] & \boxed{\Phi^{(k)}} \end{tikzcd}$$

- Length of a path P : $\ell(P) = \underline{\#\{S_i \in P\}}$
  number of steps in P

- weight of a path $\omega(P) = \prod_{i=1}^{\ell(P)} \omega(S_i)$ → product of weights

- Combinatorial factor of path P

  $CF(P) = 1/(\ell(P))!$ ← permutations of $\{S_i \in P\}$

Then

$$\text{Disc}_0\,\Phi^{(n)} = -\sum_{m\geq 1}\ \sum_{P(n\to n+m)} e^{-mAz}\,\Phi_{m+n}\,(SF)_{n\to m+n}$$

$$\text{Disc}_\pi\,\Phi^{(n)} = -\sum_{m\geq 1}^{n-1}\ \sum_{P(n\to n-m)} e^{+mAz}\,\Phi_{n-m}\,(SF)_{n\to n+m}$$

Where

$$(SF)_{n\to m} = \sum_{P(n\to m)} CF(P)\,\omega(P)$$

↓
statistical factor

From the discontinuities, one can derive the <u>Large order behaviour</u>

The $a_k^{(n)}$ for large $k \gg 1$ associated to a nodo $\Phi^{(n)}$ has large order behaviour given by sum over all forward and backward paths linking to nodes $\Phi^{(m>n)}$ and $\Phi^{(m<n)}$

▶ From Borel transforms

$$\mathrm{Disc}_0 \, \Phi^{(n)} = - \sum_{m \geq 1} S_{n \to n+m} \, e^{-mAz} \, \Phi_{m+n}$$

$$\mathrm{Disc}_\pi \, \Phi^{(n)} = - \sum_{m=1}^{n-1} S_{n \to n-m} \, e^{mAz} \, \bar{\Phi}_{n-m}$$

where

$$B[\Phi^{(n)}](t) \Big|_{t=mA} = + \frac{S_{n \to n+m}}{2\pi i} \, B[\Phi^{(n+m)}](t-mA) \log(t-mA)$$

$n+m \geq 1$

3 SUMMATIONS

3.1 Optimal truncation and hyperasymptotics

If we have our expansion (small coupling / late times) from perturbation theory we can approximate the series effectively by doing Optimal truncation

$$\Phi^{(0)}(z) = \sum_{n\geq 0} a_k^{(0)} z^{-k} \qquad a_k^{(0)} \sim \frac{\Gamma(k+\beta)}{A^{k+\beta}}$$

For each value of z we truncate $n_{op} = \lfloor |Az| \rfloor$

$\to \Phi^{(0)}_{opt} = \sum_{n=0}^{n_{op}-1} a_k^{(0)} z^{-k}$ and the Remainder, of order of first term not included

$\to R_{n_{op}} \sim e^{-A|z|}$ (least term is exp small)

Thus this gives us exponential accuracy.

But we know we have not only our perturbative series but also a full transseries

$$F(z,\sigma) = \sum_{n\geq 0} \left(\sigma e^{-zA} z^{\beta}\right)^n \Phi^{(n)}(z)$$

and including more sectors should improve our accuracy. There should be an extension of optimal truncation for exponential sectors: Hyperasymptotics

It is not enough to do optimal truncation of every sector! Berry, Howls / Olde Daalhuis constructed this extension, which can be proven by analysing the Remainder and using large order relations (all the late terms are included in R_N) to reexpand the R_N in terms of coefficients of the exponentially suppressed sectors $\Phi^{(n\geq 1)}$.

To include the $\Phi^{(1)}$ in the transseries, we need to

1. Include extra terms in the perturbative series
   $N^{(0)}_{hyp} = 2\lfloor |Az| \rfloor$ (twice optimal number)

2. Include the first exp sector in the form of a so-called hyperterminant, which not only corrects the error of including extra terms in truncation, but also decreases the overall error $\sim e^{-2A|z|}$!!

The number of terms we have from $\Phi^{(1)}$ is

$$N^{(1)}_{Hyp}(z) = \lfloor |Az| \rfloor = N^{(0)}_{Hyp}/2$$

The level-1 hyperasymptotics then approximates our solution by

$$F_{hyp}(z,\sigma) = f_{HYP,0}(z) + \sigma f_{HYP,1}(z)$$

$$f_{HYP,0}(z) = \sum_{m=0}^{N^{(0)}_{Hyp}(z)} a^{(0)}_m z^{-m}$$

Stokes constant

$$+ z^{1-N^{(0)}_{Hyp}(z)} \boxed{\frac{S_1}{2\pi i}} \sum_{m=0}^{N_{Hyp}/2-1} a^{(1)}_m F^{(1)}\left(z; \begin{matrix} N_{Hyp}+\beta-m \\ -A \end{matrix}\right)$$

$$f_{HYP,1}(z) = e^{-Az} z^{\beta} \sum_{m=0}^{N_{Hyp}/2-1} a^{(1)}_m z^{-m}$$

Where $F^{(1)}$ is hyperterminant

incomplete Gamma functions

$$F^{(1)}\left(\omega; \begin{matrix} M \\ a \end{matrix}\right) = e^{a\omega + i\pi M} \omega^{M-1} \Gamma(M) \Gamma(1-M, a\omega)$$

→ double exponential accuracy $\sim e^{-2|Az|}$

→ Functional form, but discontinuous behaviour because of jumps on the value of N_{Hyp} depending on z (special for smaller $|z|$)

3.2 Borel summation

We can also sum each sector $\Phi^{(n)}$ via Borel summation

$$S\Phi^{(n)}(z) = \int_0^{\infty} dt\, e^{-zt}\, \mathcal{B}[\Phi](t)$$

→ We don't know $\mathcal{B}[\Phi](t)$ in closed form, so we need to approximate it by e.g. Padé approximant

$$S_N\Phi^{(n)}(z) = \int_0^{\infty} dt\, e^{-zt}\, BP_N[\Phi](t)$$

if we include N terms. We can include any amount of terms we want. The approximation (at same exponential order as above) gives us

$$F_{Borel}(z,\sigma) = S_{N_0}\Phi^{(0)}(z) + \sigma e^{-Az} S_{N_1}(z^{\beta}\Phi^{(1)})(z) \overset{\substack{\text{error}\\ \downarrow}}{\underline{\sim}} \mathcal{O}(e^{-2Az})$$

Where for each z we need to perform a numerical integral

→ If we only included $S_{N_0}\Phi^{(0)}$ then taking $N_0 > N_{op}$ will not correct the error beyond e^{-Az} (although it won't make it worse either) because this error is coming from not including the second sector!
If we include $S_{N_0}\Phi^{(0)}$ and $S_{N_1}\Phi^{(1)}$, then including $N_0 > N_{Hyp}$, $N_1 > N_{Hyp/2}$ will not correct it further than e^{-2Az}, as this is the error of not including $\Phi^{(2)}$ ... (

→ Including more terms than needed will not increase the error, so less worry on optimally truncate

→ It is numerical for each value of z

→ Generalization to include extra sectors is trivial, unlike Hyperasymptotics, where formulas become very complicated.

$\boxed{3.3}$ Global behaviour and transasymptotics

$\boxed{3.3.1}$ We have our parameter σ which is yet undefined

Because of ampiguity cancelation we know we need

$$\operatorname{Im} F = 0 \Rightarrow i \operatorname{Im}\sigma = S_{1/2}$$

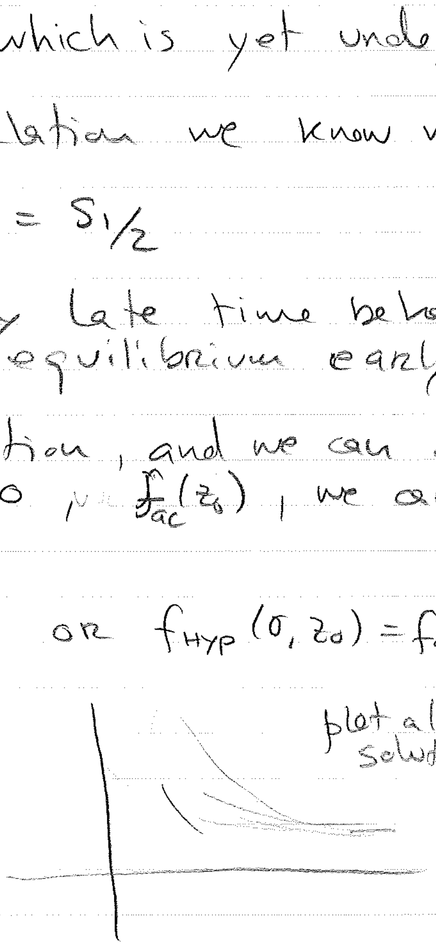
$\operatorname{Re}\sigma$ is not constrained by late time behaviour, it is related to the out of equilibrium early time.

So if we know initial condition, and we can analytically continue it to some value $z_0 > 0$, $f_{ac}(z_0)$, we can fix the $\operatorname{Re}\sigma$ by comparing

$$F_{BOREL}(\sigma, z_0) = f_{ac}(z_0) \quad \text{or} \quad f_{HYP}(\sigma, z_0) = f_{ac}(z_0)$$

$$\Rightarrow \boxed{\sigma}$$

With this value we can then determine $F(\sigma, z)$ and compare it to exact solution



plot all solutions

$\boxed{3.3.2}$ Looking at transseries we notice that we can perform a different type of sum

$$F(z,\sigma) = \sum_{n=0}^{\infty} \left(\sigma e^{-Az} z^{\beta}\right)^n \sum_{k=0}^{\infty} a_k^{(n)} z^{-k}$$

$$= \underbrace{\sum_{k=0}^{\infty} z^{-k}}_{} \; \underbrace{\sum_{n=0}^{\infty} \tau^n a_k^{(n)}}_{} \qquad \boxed{\tau = \sigma e^{-Az} z^{\beta}}$$

this is the asymptotic sum

this is convergent and can be computed exactly or to high approximation

$$F(z,\sigma) = \sum_{k=0}^{\infty} z^{-k} F_k(\tau) \qquad F_k(\tau) = \sum_{n\geq 0} \tau^n a_k^{(n)}$$

This sum is called a <u>transasymptotic sum</u> and is very effective ito study regimes where $|z| \gg 1$ (large enough) but $e^{-Az} \sim O(1)$

For our case $F_k(\tau)$ obey ODES with respect to τ.

$$F_0(\tau) = \frac{2}{3}\left(1 + W\left(3/2\, \tau\right)\right) \qquad W(x)e^{W(x)} = x \quad \text{Lambert-W}$$

$\vdots$

$$F_k(\tau) = \frac{P_k(F_0(\tau))}{Q_k(F_0(\tau))}$$

polynomials of F_0

What can we use this for? for example determining zeros / branch points at finite values of z.

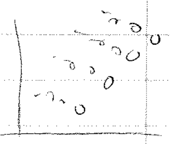
$F(z,\tau) = 0$ are $\sqrt{\ }$ - branch points related to single $\sqrt{\ }$ -branch point of $W(\tau)$, $\tau = -e^{-1}$

1. Assume τ, z independent. Solve $F(z,\tau) = 0$ for

$$\tau(z) = \tau_0 + \tau_1 z^{-1} + \tau_2 z^{-2} + \cdots$$

2. Use $\tau = \sigma e^{-Az} z^{\beta} = \tau_0 + \tau_1 z^{-1} + \tau_2 z^{-2} + \cdots$

and invert equations to obtain z



$$z_{b.p.}(t) \simeq \frac{t}{A} - \frac{\beta}{A}\log(t) + \frac{1}{At}\left(\beta^2 \log t + \beta^2 + 5\beta - \frac{3}{A}\right) \qquad t \to \infty$$

where $t = \log\left(\frac{3\sigma e}{2A^{-\beta}}\right) + \pi i\,(1+2n) \quad n \in \mathbb{Z}$! Accurate for $n \gg 1$ even with few terms!

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