Problems on Resurgence

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I Self-energy in the $\mathrm{O}(N)$ non-linear sigma model

The $\mathrm{O}(N)$ non-linear sigma model is a quantum field theory in Euclidean 2-dimensional space with N particles $\vec{\sigma} = (\sigma_1, \ldots, \sigma_N)$. The Lagrangian is given by

$$\mathcal{L}(\vec{\sigma}, X) = \frac{1}{2g^2}\Big\{\partial_\mu\vec{\sigma}\cdot\partial^\mu\vec{\sigma} + X(\vec{\sigma}^2 - 1)\Big\},$$

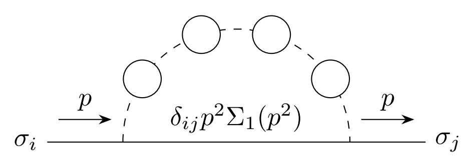
where g is the coupling of the theory and X is an auxiliary field that imposes the constraint

$$\vec{\sigma}^2 = 1.$$

The self-energy of the $\mathrm{O}(N)$ non-linear sigma model can be computed at large N in the 't Hooft coupling $\lambda = Ng^2/(2\pi)$. The result can be expanded in powers of $1/N$:

$$\Sigma(p^2) = \Sigma_0(p^2) + \frac{1}{N}\Sigma_1(p^2) + \mathcal{O}\bigg(\frac{1}{N^2}\bigg).$$

In particular, the subleading correction $\Sigma_1(p^2)$ can be obtained from diagrams of the type



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where the solid line represents the propagators of the σ_i, and the dashed line, the propagator of X. However, there is an alternative computation which exploits that the Lagrangian (**??**) is quadratic in the fields σ_i, so they can be integrated out in the path integral. This process yields an exact result, which goes beyond perturbation theory, and is given by

$$\Sigma_1(p^2) = \frac{1}{x}\int_0^\infty \mathrm{d}y \log^{-1}\bigg(\frac{\xi+1}{\xi-1}\bigg)\Bigg[\frac{y\xi}{\sqrt{(1+y+x)^2 - 4xy}} - 1 + \frac{x+1}{2}\bigg(\frac{1}{\xi} - 1\bigg)\Bigg],$$

where $x = p^2/m^2$, $m^2 = \mu^2 e^{-2/\lambda(\mu)}$ is the non-perturbatively generated mass of the σ particles, μ is the renormalization scale, and

$$\xi = \sqrt{1 + \frac{4}{y}}.$$

According to a paper by M. Beneke, V. M. Braun and N. Kivel from 1998, the self-energy (**??**) can be written as the following trans-series:

$$\Sigma_1(p^2) = \log(\lambda/2) + 1 - \gamma_E + \int_0^{\infty e^{i\theta}} e^{-y/\lambda}B_0(y)\mathrm{d}y$$
$$+ e^{-2/\lambda}\bigg(\log(\lambda/2) + 1 - \gamma_E \pm i\pi - \int_0^{\infty e^{i\theta}} e^{-y/\lambda}B_1(y)\mathrm{d}y\bigg) + \mathcal{O}\big(e^{-4/\lambda}\big),$$

where λ is the coupling at $\mu^2 = p^2$ and the two Borel transforms are given by

$$\begin{aligned} B_0(y) &= \frac{2}{y-2} - \frac{1}{2}\psi(1+y/2) - \frac{1}{2}\psi(1-y/2) - \gamma_E, \\ B_1(y) &= \frac{1}{2} - \frac{y}{2} + \frac{y^2}{4} + \frac{1}{2}\left(1 - \frac{y^2}{4}\right)\left[\psi(1+y/2) + \psi(2-y/2) + 2\gamma_E\right]. \end{aligned}$$

γ_E is the Euler's constant and $\psi(x) = \mathrm{d}\log\Gamma(x)/\mathrm{d}x$ is the digamma function. The ambiguous imaginary signs in the exponential correction are chosen according to the standard convention: the upper sign has to be paired with a direction $\theta > 0$ in the integration, while the lower sign has to be paired with a direction $\theta < 0$.

1.

(a) Recover the full perturbative expansion of the self-energy, and obtain

$$\Sigma_1(p^2) \sim \log(\lambda/2) + 1 - \gamma_E - 2\sum_{k\geq 0} k!\,(\lambda/2)^{k+1} + 2\sum_{k\geq 1}(2k)!\,\zeta(2k+1)(\lambda/2)^{2k+1},$$

where $\zeta(z)$ is the Riemann zeta function. Remember that the Taylor expansion of the digamma function around $1 + z = 0$ is given by

$$\psi(1+z) = -\gamma_E - \sum_{k\geq 1}\zeta(k+1)(-z)^k.$$

Is the expansion (**??**) convergent for small enough λ?

(b) Consider the coefficients c_k of the asymptotic expansion of (**??**), defined as

$$\sum_{k\geq 0} c_k \lambda^{k+1}.$$

Check that for large and odd k, the coefficients behave as

$$c_k \sim C\, k!\, (1/A)^k,$$

where C and A are real constants. What is the value of A?

(c) The integral in (**??**) can be numerically computed for specific values of x. In particular, for $x = 1000$, we obtain

$$\Sigma_1(m^2 x) \approx -1.848040909962.$$

Approximate the self-energy by truncating the asymptotic expansion of (**??**). Remember that $x = e^{2/\lambda}$. Consider truncation to order λ^{k_0+1} with k_0 odd integers in the interval $1 \leq k_0 \leq 21$. Plot the approximations as a function of k_0. Are they a good approximation to the numerical result of (**??**)? What is the value of k_0 that yields the best approximation? Compare the optimal k_0 with the theoretical expectation

$$k_0 \approx \frac{A}{\lambda}.$$

2.

(a) Check that the imaginary ambiguity arising from the Borel sum of the perturbative part cancels with the imaginary ambiguity in the exponential corrections, up to order $e^{-2/\lambda}$ (included). Do it both numerically and analytically.

(b) What should be the imaginary ambiguity in the $e^{-4/\lambda}$ correction, so that it cancels the imaginary ambiguity arising from both the Borel sum of the perturbative part, and the borel sum of the $e^{-2/\lambda}$ exponential correction?

(c) Numerically compute the Borel sum of the perturbative part at $x = 1000$. Repeat the analysis of exercise 1, but for the quantity

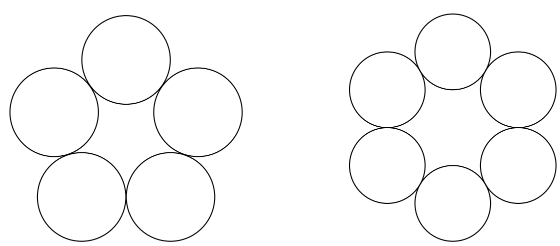
$$\Sigma_1(p^2) - \left[\log(\lambda/2) + 1 - \gamma_E + \mathrm{Re}\int_0^{\infty e^{i\theta}} e^{-y/\lambda} B_0(y)\mathrm{d}y\right].$$

II Ground state energy density in the O(N) sigma model with a quartic potential

The O(N) sigma model with a quartic potential is a quantum field theory of N scalar particles $\vec{\Phi} = (\Phi_1, \ldots \Phi_N)$. The Lagrangian is given by

$$\mathcal{L}(\vec{\Phi}) = \frac{1}{2}\partial_\mu\vec{\Phi}\cdot\partial^\mu\vec{\Phi} - \frac{\mu^2}{2}\vec{\Phi}^2 - \frac{g}{4!}\vec{\Phi}^4.$$

At large N, the ground state energy density of this model can be computed in perturbation theory from diagrams of the type (called "ring diagrams")



where the solid lines denote the propagators of the Φ_i particles.

According to a paper by M. Mariño and T. Reis from 2020, the ground state energy density E of this model, at large N and at the perturbative level, is given by

$$\frac{E(\gamma)}{m^2} = -\frac{N}{32\pi\gamma} + \frac{1}{8\pi} - \frac{\pi}{48}\gamma - \frac{I(\gamma)}{8\pi} + \mathcal{O}\left(e^{-\frac{1}{\gamma}}\right) + \mathcal{O}(N^{-1}),$$

where $\gamma = Ng/(12\pi m^2)$ is a dimensionless 't Hooft coupling, $m^2 = -2\mu^2$, and

$$I(\gamma) = \int_0^\infty \left\{\log\left[1 + \gamma\frac{\log(x)}{x+1}\right] - \gamma\frac{\log(x)}{x+1}\right\}\mathrm{d}x$$

3.

(a) Take the N^0 contribution from the above result and expand it in powers of the coupling γ. You should obtain

$$\frac{1}{8\pi} - \frac{\pi}{48}\gamma - \frac{1}{8\pi}\sum_{n\geq 2} c_n \gamma^n, \qquad c_n = \frac{(-1)^{n+1}}{n}\int_0^\infty \left[\frac{\log(x)}{x+1}\right]^n \mathrm{d}x.$$

(b) Numerically compute the coefficients c_n for $2 \leq n \leq 50$. Check that $c_n/(n-1)! \to -1$ as n becomes large. What is the large order behavior of the coefficients?

4. In this exercise, we will analytically derive the large order behavior that we computed above. First, notice that the integral $I(\gamma)$ has an imaginary part, which arises from the region of integration where $1 + \gamma \log(x)/(x+1) < 0$. In particular, this region is given by $0 < x < x(\gamma)$, where

$$1 + \gamma \frac{\log(x(\gamma))}{x(\gamma)+1} = 0.$$

(a) For $y > 0$, an imaginary part arises from the logarithm: $\log(-y \pm i0) = \log(y) \pm i\pi$. Check that the total imaginary part in $I(\gamma \pm i0)$ is given by

$$\mp i\pi x(\gamma).$$

Assume $0 < x(\gamma) < 1$, so $\log(x(\gamma)) < 0$.

(b) We note that (**??**) is a transcendental equation for $x(\gamma)$ and it cannot be solved by algebraic methods. The Lambert's function $W(z)$ is defined as

$$W(z)e^{W(z)} = z,$$

providing a solution w to the transcendental equation $we^w = z$. Prove that (**??**) can be written as

$$\frac{x(\gamma)}{\gamma}e^{x(\gamma)/\gamma} = \frac{e^{-1/\gamma}}{\gamma}.$$

Then verify that

$$x(\gamma) = \gamma W\left(\frac{e^{-1/\gamma}}{\gamma}\right).$$

(c) The power expansion of the Lambert's function around $z = 0$ is given by

$$W(z) = \sum_{n\geq 1} \frac{(-n)^{n-1}}{n!} z^n.$$

Write the imaginary ambiguity of $I(\gamma \pm i0)$ as a sum of exponentially small terms in the coupling γ, and obtain, to the very first terms

$$\mp i\pi\Big(e^{-1/\gamma} - e^{-2/\gamma}\gamma^{-1} + \mathcal{O}\big(e^{-3/\gamma}\big)\Big).$$

(d) Assume that

$$I(\gamma \pm i0) = \frac{1}{\gamma}\int_0^{\infty e^{i\theta_\pm}} e^{-s/\gamma} B(s)\mathrm{d}s,$$

where $B(s) = \sum_{n\geq 2} c_n s^n/n!$ is the Borel transform of the asymptotic expansion of $I(\gamma)$, and θ_+ (θ_-) is a small positive (negative) angle. From the ambiguity in (**??**), compute the large order behavior of the coefficients c_n up to two asymptotic terms.

III Self-energy in massless QED

In QED, the photon propagator is given by

$$iD_{\mu\nu}(q) = \frac{i}{q^2}\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right],$$

the propagator for massless leptons is

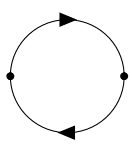
$$iS_{ij}(p) = i\delta_{ij}\frac{\not{p}}{p^2},$$

and the interaction vertex is $ie_0\gamma^\mu\delta_{ij}$, where e_0 denotes the bare QED coupling.

In this list of exercices, we will compute the self-energy of the leptons for massless QED in the limit of large number of leptons N, but at all orders in perturbation theory. To this end, we define the 't Hooft coupling

$$\lambda_0 = \frac{Ne_0^2}{12\pi^2}.$$

We recall that the fermion polarization function is given by

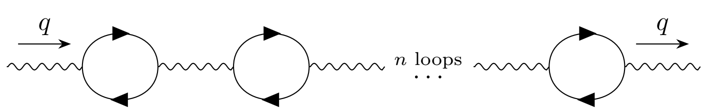
$i\lambda_0\Pi_{\mu\nu}(q) =$ $ie_0\gamma_\mu$  $ie_0\gamma_\nu$ $= i\lambda_0(q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2),$

with ($d = 4 + 2\epsilon$ and $(+, -, \cdots, -)$ Minkowski metric)

$$\Pi(q^2) = \frac{12\pi^2 i}{q^2(1-d)}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{\mathrm{Tr}[\gamma^\mu(\not{p}+\not{q})\gamma_\mu\not{p}]}{p^2(p+q)^2} = \left(-\frac{p^2}{4\pi}\right)^\epsilon\frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)\Gamma(2+\epsilon)}{\epsilon(1+2\epsilon/3)\Gamma(2\epsilon+2)}.$$

5.

(a) Construct a chain of n lepton loops joined together by $n+1$ photon propagators



$$[iD_{\mu_1\mu_2}(q)][i\lambda_0\Pi^{\mu_2\mu_3}(q)][iD_{\mu_3\mu_4}(q)]\cdots[iD_{\mu_{n+1}\mu_{n+2}}(q)].$$

Check that

$$\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\left(-g^{\nu\rho} + \frac{q^\nu q^\rho}{q^2}\right) = -\left(-g_\mu^\rho + \frac{q_\mu q^\rho}{q^2}\right).$$

Conclude from this result that the chain of lepton loops can be expressed as

$$-\frac{i}{q^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\Pi(q^2)^n\lambda_0^n.$$

(b) Add an extra factor λ_0 to the above result, to account for the vertices at each end of the chain, and sum the chains to obtain (up to factors in front of the sum)

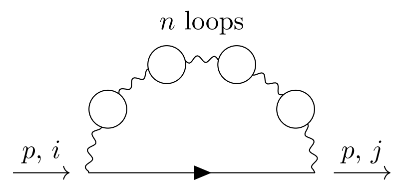
$$\lambda_0 \sum_{n\geq 0} \Pi(q^2)^n \lambda_0^n = \frac{\lambda_0}{1-\lambda_0 \Pi(q^2)}.$$

(c) We consider the renormalized coupling $\lambda = (\nu^2)^\epsilon Z_\lambda \lambda_0$, where Z_λ is the renormalization coupling constant and $\nu^2 = \mu^2 e^{\gamma_E - \log(4\pi)}$ accounts for both the renormalization scale μ and the modified minimal subtraction scheme. Obtain

$$Z_\lambda^{-1} = \frac{1}{1+\lambda/\epsilon} + \mathcal{O}(1/N).$$

by renormalizing the chain of lepton loops.

6. Draw a big loop formed with a chain of n lepton loops and one lepton propagator, and obtain the $(n+1)$-th perturbative correction to the bare self-energy:



$$\frac{\delta_{ij}}{N} 12\pi^2 i \lambda_0^{n+1} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{-i}{q^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\Pi(q^2)^n \gamma^\mu \frac{i(\not p + \not q)}{(p+q)^2}\gamma^\nu.$$

(Do not forget to add the two extra QED vertices and multiply by $-i$ so that the self-energy is real).

(a) Using gamma matrix identities in d dimensions and $2p\cdot q = (p+q)^2 - p^2 - q^2$, write the product of gamma matrices inside the loop as:

$$\frac{1}{q^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\gamma^\mu \frac{\not p + \not q}{(p+q)^2}\gamma^\nu = \frac{(d-2)(\not p + \not q) - \not p}{q^2(p+q)^2} - \frac{p^2 \not q}{q^4(p+q)^2} + \frac{\not q}{q^4}.$$

(Notice that the last term will vanish in the loop integral, due to the symmetry $q_\mu \mapsto -q_\mu$).

(b) Prove that the bare self-energy can be written as

$$\Sigma_{ij}(p) = \not p \frac{\delta_{ij}}{N} \sum_{n\geq 1} \Sigma^{(n)}(p^2)\lambda_0^n$$

with

$$\Sigma^{(n)}(p^2) = 12\pi^2 i\left(-\frac{1}{4\pi}\right)^{(n-1)\epsilon}\left[\frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)\Gamma(2+\epsilon)}{\epsilon(1+2\epsilon/3)\Gamma(2\epsilon+2)}\right]^{n-1}$$
$$\times \big[(1+2\epsilon)I(1-(n-1)\epsilon, 1) + (2+2\epsilon)J(1-(n-1)\epsilon,1) - p^2 J(2-(n-1)\epsilon, 1)\big].$$

The loop integrals I and J are defined as

$$I(r,s) = \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(q^2)^r[(p+q)^2]^s}, \qquad p^\mu J(r,s) = \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{q^\mu}{(q^2)^r[(p+q)^2]^s}.$$

7. Consider a bare Green's function expanded in powers of the bare coupling:

$$G_0(\epsilon) = \sum_{n\geq 1} G_0^{(n)}(\epsilon)\lambda_0(\epsilon)^n.$$

We define the "structure function" $F(x, y)$ of the Green's function through

$$(\nu^2)^{-n\epsilon} G_0^{(n)}(\epsilon) = \frac{1}{n\epsilon^n} F(\epsilon, n\epsilon),$$

We also define the expansion of F in the second variable as

$$F(x, y) = \sum_{j\geq 0} F_j(x) y^j.$$

(a) Using the renormalization constant of (**??**), prove that the Green's function can be expanded in terms of the renormalized coupling as

$$G_0(\epsilon) = -F_0(\epsilon)\sum_{n\geq 1}\frac{1}{n}\left(-\frac{\lambda}{\epsilon}\right)^n + \sum_{n\geq 1}(n-1)!F_n(\epsilon)\lambda^n + \mathcal{O}(\epsilon).$$

Indication: Use the Taylor expansion

$$\left(\frac{x}{1+x}\right)^n = \sum_{m\geq n}\binom{-n}{m-n}x^m,$$

with $x = \lambda_0/\epsilon$, then commute the sum over n with the sum over m. Finally, use the binomial identity

$$\sum_{n=1}^{m}\binom{-n}{m-n}n^{j-1} = \begin{cases}(-1)^{m+1}/m & \text{if } j = 0,\\ 0 & \text{if } 1 \leq j \leq m-1,\\ (m-1)! & \text{if } j = m.\end{cases}$$

(b) After the introduction the renormalization field constant for the leptons, we can renormalize the Green's function by simply removing the first sum of (**??**) and then taking the limit $\epsilon \to 0$. Prove that the renormalized Green's function can be written as the Borel sum

$$G(\lambda) = \int_0^\infty e^{-t/\lambda}B(y)\mathrm{d}y,$$

where B is the Borel transform

$$B(y) = \frac{F(0, y) - F(0, 0)}{y}.$$

Assume that the functions $F_n(x)$ are convergent at $x = 0$.

8.

(a) With the help of an algebraic program, if necessary, compute the structure function of the lepton self-energy, using the results of (**??**) and (**??**). Use the following expressions for the loop integrals I and J (they can be found, for example, in the book "QCD: Renormalization for the Practitioner," by P. Pascual and R. Tarrach, appendix C):

$$
\begin{aligned}
I(r,s) &= \frac{i}{(4\pi)^2}\left(-\frac{p^2}{4\pi}\right)^{\epsilon}\frac{1}{(p^2)^{r+s-2}}\frac{\Gamma(2-r+\epsilon)\Gamma(2-s+\epsilon)\Gamma(r+s-2-\epsilon)}{\Gamma(r)\Gamma(s)\Gamma(4-r-s+2\epsilon)}, \\
J(r,s) &= -\frac{i}{(4\pi)^2}\left(-\frac{p^2}{4\pi}\right)^{\epsilon}\frac{1}{(p^2)^{r+s-2}}\frac{\Gamma(3-r+\epsilon)\Gamma(2-s+\epsilon)\Gamma(r+s-2-\epsilon)}{\Gamma(r)\Gamma(s)\Gamma(5-r-s+2\epsilon)}.
\end{aligned}
$$

Write the structure function in terms of the variables $x=\epsilon$ and $y=n\epsilon$ (when n appears alone, use $n=y/x$).

(b) Compute the Borel transform of the renormalized self-energy, and obtain

$$
B(y) = \frac{9}{4}\left(-\frac{p^2}{\mu^2}\right)^{y}\frac{e^{-5y/3}}{(y+2)(y+1)(y-1)}.
$$

Draw a qualitative plot with the position of the singularities.

(c) Consider the scale choice $\mu^2 = -p^2 > 0$. We define the perturbative coefficients c_n as

$$
\Sigma(\lambda) = \sum_{n\geq 1} c_n \lambda^n.
$$

From the Borel transform you computed above, determine the large order behavior of the coefficients c_n, arising from the closest singularity (or singularities) to the origin.