Riemann Surfaces & Theta functions

There are different ways of thinking about a RS, depending on context/use

In this course "Riemann Surface" will imply "compact" and "smooth"

At its core it is a 2-dim (real) oriented manifold (hence orientable)

Examples

Non compact

$\mathbb{C} \simeq \mathbb{R}^2$

$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$

$\mathbb{H}^+ = \{ w \in \mathbb{C} : \Im m\, w > 0 \}$

Compact

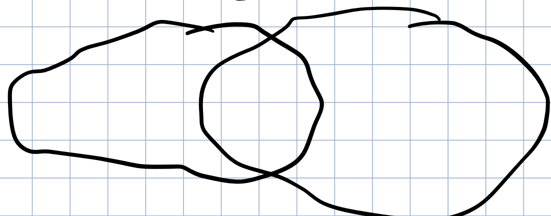
- $S^2 = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \}$
-  Torus

$S^1 \times S^1$

Assuming that the notion of manifold (set $\circ$ atlas of charts) is known, then

Def A R.S. is a 2-dim manifold with a set of charts such that, in each intersection

$z = x+iy$ $\qquad$ $w = u+iv$



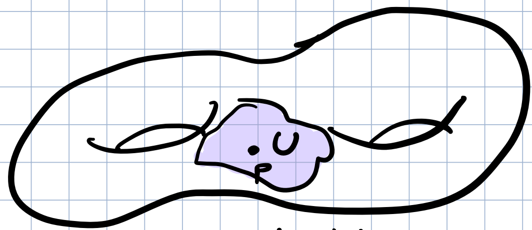
$w(z) = u(x,y) + i v(x,y)$

is holomorphic

$\frac{dw}{dz} \neq 0$. Ditto for the inverse

"complex (or holomorphic) structure"

In practice: A R. S. is a 2-dim mfld with a rule (often "unspoken") to assign a local holomorphic complex coordinate to the neighbourhood of every point



$z : U \mapsto \mathbb{C}$

so that the rule is "consistent" in each overlap. "Consistent" as above

Example:

① On $\mathbb{C}$ or $\mathbb{D}$ or $\mathbb{H}^+$ we can cover all with the tautological but nothing prevents us from choosing a different local coordinate

② The Riemann sphere: $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$

Two open charts U_0, $U_\infty = \mathbb{P}^1 \setminus \{0\}$
U_0 "=" $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$

on U_0 we use z (tautologically), on U_∞ we use w such that

$w(z) = \frac{1}{z}$ on the overlap and $w(\infty) := 0$.

At this point the rule is that in any nbd we can choose any holomorphic function φ of z (or w) as long as

① $\varphi' \neq 0$ & ② φ is invertible (where defined) with holomorphic inverse

Lesson: It is enough to define a consistent choice of finitely many open charts/coordinates.

Example (Weierstrass)

- If $f: \mathcal{D}\subset\mathbb{C}\to\mathbb{C}$ is holomorphic then, $0\in\mathcal{D}$

$$\mathcal{R}_f = \left\{ (z,w) \in \mathbb{C}^2 : w = f(z) \text{ for } \underline{\text{some}} \text{ analytic continuation of } f \text{ from } 0 \text{ to } z \right\}$$

In this case the local coordinate is given by $z=\pi(p)$ (projection onto first factor)

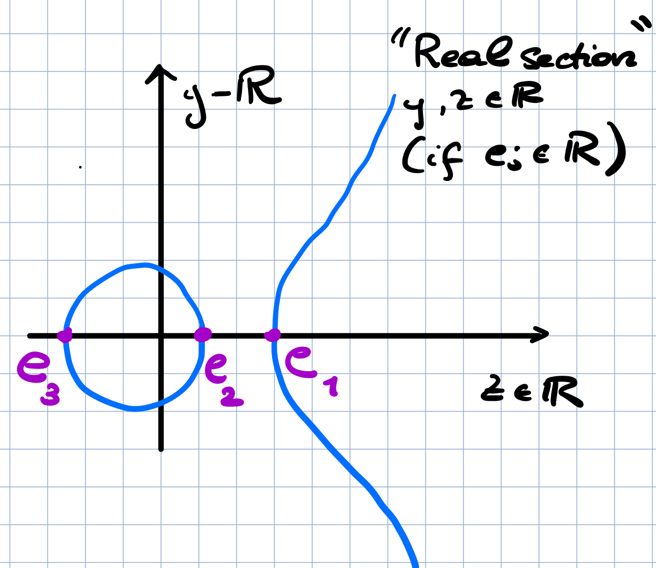
Example

$$\mathcal{C} = \left\{(z,y)\in\mathbb{C}^2 : y^2=(z-e_1)(z-e_2)(z-e_3)\right\}$$

We **can** but it is not always the best think of it as the Weierstrass R.S. of

$$\sqrt{(z-e_1)(z-e_2)(z-e_3)}$$

(you need to specify branches & branch-cuts)



Local coordinates

Near $(e_j, 0)$ we have "vertical tangent": we **cannot** use z for coordinate:

Standard choice is to use y or $\zeta = \sqrt{z - e_1}$ (e.g.)

$(z = \zeta^2 + e_1)$

Check: $$y = \sqrt{(z-e_1)(z-e_2)(z-e_3)}$$

$y = \zeta \cdot \sqrt{(\zeta^2 + (e_1 - e_2))(\zeta^2 + (e_1 - e_3))}$ is analytic for $\zeta \simeq 0$.

Example (plane algebraic curves) $\longrightarrow$ Recurring example.

Given a polynomial $P(z,y) = 0$

$$\mathcal{C} = \{ (z,y) \in \mathbb{C}^2 : P(z,y) = 0 \}$$

(more generally one can have $\left\{ (z,y,w) \in \mathbb{C}^3 : \begin{matrix} P_1(z,y,w) = 0 \\ P_2(z,y,w) = 0 \end{matrix} \right\}$ etc. ("complete intersections")

Def $\mathcal{C}$ is smooth if $\begin{cases} P_z(z,y) = 0 \\ P_y(z,y) = 0 \\ P(z,y) = 0 \end{cases}$ has no solutions

Note: one can classify the singularities: the simplest is the **node** locally modelled on $z^2 - y^2 = 0$

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If $\mathcal{C}$ is smooth, in the ngbd of any $(z_0,y_0)\in\mathcal{C}$ we have either $P_z\neq 0$ or $P_y\neq 0$ (or both)

A local coord is

$\zeta=(z-z_0)$ if $P_y(z_0,y_0)\neq 0$ or $\zeta=(y-y_0)$ if $P_z(z_0,y_0)\neq 0$ (or any $\varphi(\zeta)$ holomorphic, $\varphi'(0)\neq 0$)

Example $P = y^2-(z-e_1)(z-e_2)(z-e_3)=0$

$P_y = 2y$ $\qquad$ $P_z = \cdots$

near $(z_0,y_0)=(e_1,0)$ we must use $y-y_0=y$ because $P_y(z_0,y_0)=0$

Exercise Show that if $e_i\neq e_j$ for $i\neq j$ then

$$y^2=\prod_{\ell=1}^{2g+1}(z-e_\ell) \quad \text{is } \underline{\text{smooth}}$$

Hyperelliptic curve

Exercise Find values of a for which this curve is not smooth

$$\frac{y^3}{3}-azy-\frac{z^2}{2}+1=0$$

$P_y = y^2-az$ $\qquad$ $P_z = ay - z$

$z=ay$

$3y^2-a^2y=0$ $\qquad$ $y[y-a^2]=0$

$\boxed{a\neq 0}$ $\quad$ $y=0$ $\quad z=0$ $\quad$ not

$y=a^2$ $\quad z=a^3$ $\quad\to\quad \frac{a^6}{3}-a^6-\frac{a^6}{2}+1=0$

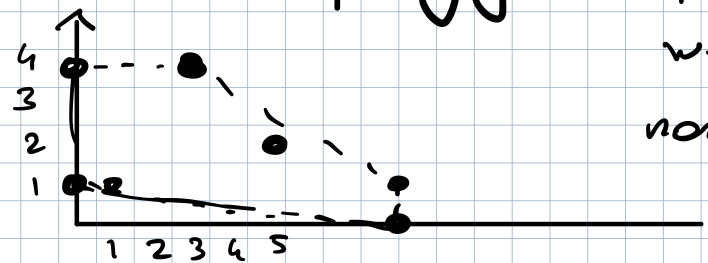
$\boxed{a=0}$ $\quad (P_y=0\Rightarrow y=0)(P_z=0\Rightarrow z=0)$ no problem.

Compactification(s)

For plane alg. curves we need to add points at "∞" (usually more than one) to (sometimes) complete to a smooth alg. curve. There is a procedure to know how many such pts (Brieskorn, page 370 & ff)

based on Newton polygon: put a dot in $(k,\ell)\in\mathbb{Z}^2$ where there is a nonzero coeff of $z^k y^\ell$ in $P(z,y)$

$$P=\sum_{i,j\geq 0} a_{ij}\, z^i y^j$$



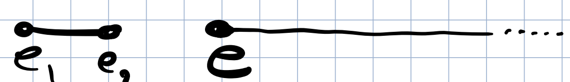
Here we only point out that this algorithm exists

Example of compactification

•) $y^2 = P_{2g+1}(z)$ (distinct roots) $\to$ one point

∞ ; local coordinate $\zeta = \frac{1}{\sqrt{z}}$

The same as embedding in $\mathbb{CP}^2$ (only for elliptic)



••) $y^2 = P_{2g+2}(z)$ (distinct roots) . ∞_+, ∞_- (two points)

local coordinate $\zeta = \frac{1}{z}$

Not the same as embedding in $\mathbb{CP}^2$

(why two points? Because there is a function that separates them:

$\frac{y}{z^{g+1}} \to \pm 1$ depending on the sheet of $\sqrt{P_{2g+2}}$

Meromorphic functions & maps

Let C be a R.S. ; a function f is holomorphic on C if, when written in a local coordinate (hence in any loc. coord.) is a holomorphic. It is meromorphic if the only singularities are poles of finite order.

Def $\mathrm{ord}_f(p) = k \in \mathbb{Z}$ where $k>0$ if $p \in C$ is a zero of order k

$k<0$ if $p \in C$ is a pole order $|k|$.

Def A divisor is a formal writing $D = \sum_{p \in C} k_p (p)$

with $k_p \in \mathbb{Z}$ and finitely many $k_p \neq 0$; $\deg D = \sum k_p \in \mathbb{Z}$

Def The divisor of a meromorphic function f is $\operatorname{div}(f) = \sum_{p\in\mathcal{C}} \operatorname{ord}_f(p)\,(p)$.

Ex: $\mathcal{C} = \{ y^2 = P_{2g+1}(z)\} \cup \{\infty\}$

z has $\underline{2}$ simple zeros $\left(0^{\pm}, \sqrt{P_{2g+1}(0)}\right)$ if $P(0)\neq 0$)

1 double pole at ∞ $z = \frac{1}{\zeta^2}$ ← local coordinate

then $\operatorname{div}(z) = (0_+) + (0_-) - 2(\infty)$

Let $\mathcal{C}$, $\mathcal{Q}$ be two R.S.

Def A map $\varphi : \mathcal{C} \mapsto \mathcal{Q}$ of 2-dim mflds is holomorphic if it is represented by a holomorphic function when represented in a local coord. near $p_0 \in \mathcal{C}$, $q_0 \in \mathcal{Q}$.

A (a.k.a. critical) ramification point of φ is a pt. $p_0 \in \mathcal{C}$ s.t. in a local coord $z(p)$ s.t. $z(p_0)=0$ and w at $q_0 \in \mathcal{Q}$

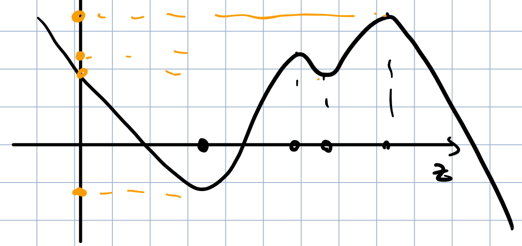
$$w(\varphi(p)) = c\cdot z^{b+1}\cdot(1+\mathcal{O}(z)) \qquad c\neq 0,\ b\geq 1$$

($\varphi(p_0)$ is called a.k.a. critical value branch point). $b = b_\varphi(p_0)$ is the ramification number.

Ex: $\mathcal{C} = \mathbb{C} = \mathcal{Q}$

$w = P(z) \Rightarrow$ critical pts. are where $P'(z)=0$

$w\uparrow$ vals " " $P(z_j) = w_j$, $P'(z_j)=0$.



Def The sheet number or degree of $\varphi : \mathbb{C} \to Q$ is the number of preimages of a generic $q \in Q$.

$$N_\varphi = \# \varphi^{-1}(\{q\}) \quad (\text{generically})$$

In general we need to count the preimages with multiplicity and the formula becomes

$$N_\varphi = \sum_{p \in \varphi^{-1}(\{q\})} (b_\varphi(p)+1)$$

Note: it does not depend on $q \in Q$.

Ex: $w = P_n(z)$ has degree n: $\forall w_0 \in Q$ there are n roots (counted with multiplicity) of the equation $P_n(z) = w_0$.

Remark Any meromorphic function on $\mathbb{C}$ can be viewed as a holomorphic map to the $\mathbb{P}^1$ where poles are mapped to ∞.
Poles of order $k \geq 2$ are ramification pts of $b = k-1$.
(Simple poles are not ram. pts.)

Ex. when a pole of a merom. function is a ramification point?

Exercise $\mathcal{C} = \{ y^2 = z(z-1)(z-\lambda) \} \cup \{\infty\}$

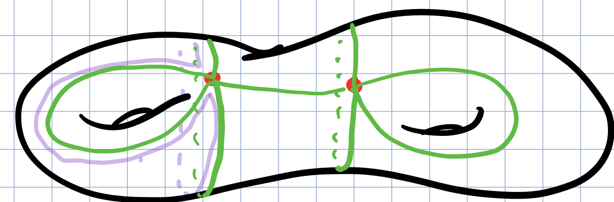
$\varphi = \dfrac{(z^2+4)y}{y-1}$ as a map $\mathcal{C} \longmapsto \mathbb{P}^1$ (see remark)

Find: ① $\varphi^{-1}(\{0\})$; ② ramification number $\forall p \in \varphi^{-1}(\{0\})$

③ degree of the map.

Triangulations and Euler characteristic.

Let S be a surface (oriented); we do not care about complex structure.



An embedded graph $\mathcal{G}$ is a collection of vertices $\mathbb{V} = \{p_1 \dots p_N\}$ connected by edges $\mathbb{E}$ s.t. each edge is a smooth simple arc e connecting two vertices (possibly the same), and mutually non intersecting.

The faces $\mathbb{F}$ are the connected components of $\mathcal{C} \setminus \mathcal{G}$.

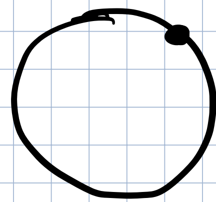
A graph $\mathcal{G}$ defines a cellularization if each face $f \in \mathbb{F}$ is simply connected.

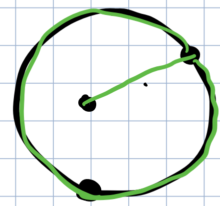
Def $V := \#\mathbb{V}$, $E := \#\mathbb{E}$, $F := \#\mathbb{F}$ Can be defined for R.S. with holes as well. (pts. or disks removed)

$$\chi(\mathcal{C}) = V - E + F$$ is the Euler characteristic.

it does not depend on the cellularization (an example of "index theorem")

Examples (1) Disk $\mathbb{D}$



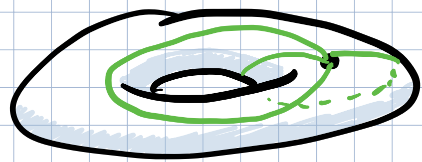


$V = 1$, $E = 1$, $F = 1$

$\chi = 1$

$V = 3$ $E = 3$, $F = 1$

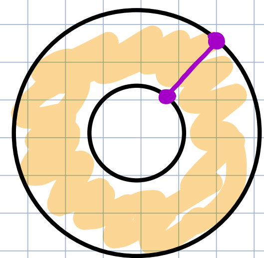
(2) Torus



$V = 1$ $E = 2$ $F = 1$

$\chi = 0$

(3) Annulus

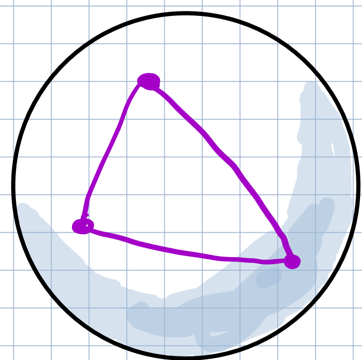


$V = 2$, $E = 3$, $F = 1$

$2 - 3 + 1 = 0$

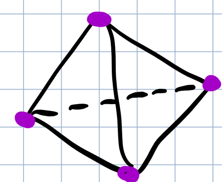
(4) Sphere

(or polyhedra)



$V = 3$ $E = 3$, $F = 2$

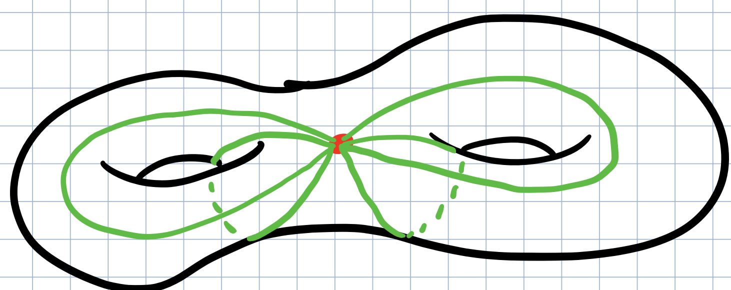
$\chi(\mathbb{P}^1) = 2$



(tetrahedron)

$V = 4$
$E = 6$ $\chi = 2$
$F = 4$

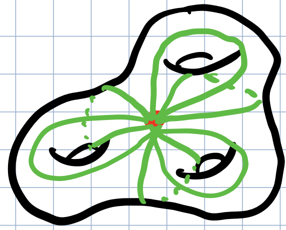
(5) Bidonut:



$V = 1$
$E = 4$
$F = 1$

$\chi = -2$

⑥ Tridonut



$V=1 \quad E=6 \quad F=1$

$\chi = -4$

Fact the Euler char is a "topological invariant": if $\mathcal{C}$ and $\tilde{\mathcal{C}}$ admit a continuous bijection, then they have the same χ. (Exercise (not entirely fair))

Elliptic curves & functions — Skip (?)

Motivation $\mathcal{C} = \{ y^2 + z^2 = 1 \} \cup \{ \infty_+, \infty_- \}$ The "real section" $(z, y \in \mathbb{R})$ is just the circle.

Exercise $\mathcal{C} \leftrightarrow \mathbb{P}^1$ birationally

Solution: this is the map. in terms of (affine) coordinate $t \in \mathbb{P}^1$

$$\begin{cases} z = \frac{1}{2}\left(t + \frac{1}{t}\right) \\ y = \frac{1}{2i}\left(t - \frac{1}{t}\right) \end{cases}$$

; the inverse map is $t = z + iy$ (z, y not real!)

$t^{-1} = z - iy$

$\mathbb{P}^1 \to \mathcal{C}$
(z, y)

Let $(S, C) \in \mathcal{C}$. (i.e. $S^2 + C^2 = 1$) and define

$$v = v((S,C)) = i \int_{(0,1)}^{(S,C)} \frac{dz}{y} \;\text{"="}\; \int_0^S \frac{dz}{\sqrt{1-z^2}} = \arcsin(S)$$

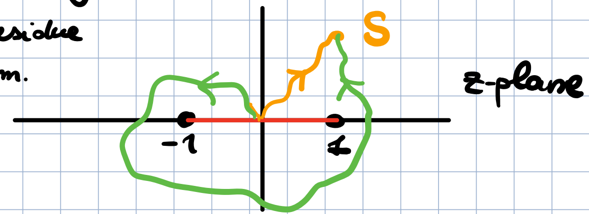
Namely: $S = \sin(v)$ (and $C = \cos(v)$) ($v \in \mathbb{C}$!)

$(\sin(v))^2 + (\cos(v))^2 = 1 \longleftrightarrow$ uniformization with $v \in \mathbb{C} \bmod 2\pi$

$v(S,C)$ is <u>not</u> single-valued: analytic continuation around ± 1 (green vs orange contours)

Cauchy residue theorem.

$$v(S,C) = v(S,C) + 2\pi$$



Elliptic curves   Skip (?)

$$\mathcal{C} = \{ y^2 = P_3(z) \} \cup \{\infty\} \quad \text{or} \quad \tilde{\mathcal{C}} = \{ y^2 = P_4(z) \} \cup \{\infty_+, \infty_-\}.$$

It is possible to biholomorphically map $\tilde{\mathcal{C}}$ to something of the form $\mathcal{C}$ (see notes). Up to some arbitrary choices we can always cast them in Weierstraß form

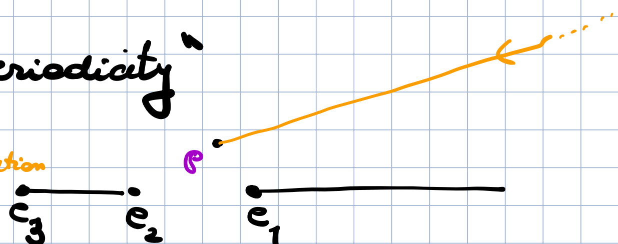
$$Y^2 = 4X^3 - g_2 X - g_3 = 4(X-e_1)(X-e_2)(X-e_3)$$
$$(e_1 + e_2 + e_3 = 0)$$

Question: can we find "sin, cos" for elliptic curves?

Answer: Yes, but now there is a "double periodicity"

$$v := \int_\infty^{\wp} \frac{dX}{Y}$$

The inverse function $\wp(v)$ → Weierstraß $\wp$-function

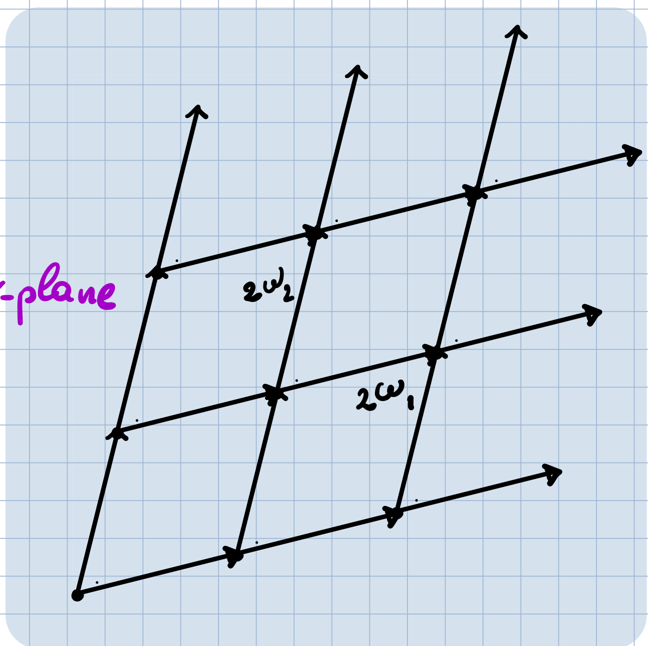


$$\wp(v) = \frac{1}{v^2} + \sum_{(\ell,k)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{(v+2\ell\omega_1+2k\omega_2)^2} - \frac{1}{(2\ell\omega_1+2k\omega_2)^2}\right]$$

where $\omega_1 = \int_{e_3}^{e_2} \frac{dX}{Y}$; $\omega_2 = \int_{e_2}^{e_1} \frac{dX}{Y}$

Fact: •) ω_1 and ω_2 are linearly indep't as vectors in $\mathbb{R}^2$ (over $\mathbb{R}$)

v-plane



(••) The series for $\wp$ converges to a meromorphic function of v with double poles at the vertices of the lattice $2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$

$$\wp(v+2\omega_1) \equiv \wp(v+2\omega_2) \equiv \wp(v)$$

(exercise!)

Doubly periodic.

Def Any {mero/holo}morphic function $f(v)$ with this double periodicity is called **elliptic**

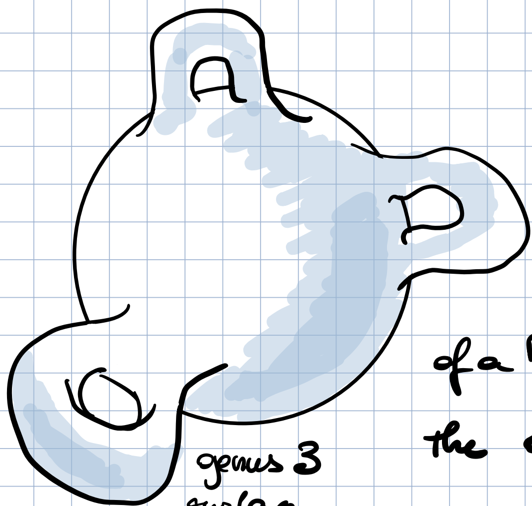
Prop: $(\wp')^2 = 4\wp(v)^3 - g_2\wp(v) - g_3$ $\quad (Y^2 = 4X^3 - g_2X - g_3)$

with $g_2 = 60 \sum'_{(\ell,k)\neq(0,0)} \frac{1}{(2\ell\omega_1+2k\omega_2)^4}$ $\quad g_3 = 140 \sum'_{(\ell,k)\neq(0,0)} \frac{1}{(2\ell\omega_1+2k\omega_2)^6}$

Lesson $\mathbb{C}$ is diffeomorphic to the parallelogram with identified opposite sides: i.e. a **torus.**

Topology and differential calculus.

Theorem: any compact oriented surface is diffeomorphic to a "sphere with handles": the **genus** of the surface is the number of handles. The sphere has genus 0.



genus 3 surface

Remark The genus is another topological invariant.

Def The fundamental group $\pi_1(C, p_0)$ of a R.S. C with basepoint $p_0 \in C$ is the group whose elements are equivalence classes of contours starting and ending at p_0, with the equivalence given by homotopy, and group multiplication given by **concatenation**.

(continuous)

$$\gamma : [0,1] \to C, \quad \gamma(0) = \gamma(1) = p_0 \qquad \eta : [0,1] \to C \; ; \; \eta(0) = \eta(1) = p_0$$

concatenation

- $\gamma \odot \eta : [0,1] \to C$ defined $\gamma \circ \eta(t) = \begin{cases} \eta(2t) & t \in [0, \frac{1}{2}) \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$

••) $\gamma \sim \tilde{\gamma}$ if $\exists\, F : [0,1]\times[0,1] \to \mathbb{C}$ (continuous) s.t. $\begin{cases} F(t,0) = \gamma(t) \\ F(t,1) = \tilde{\gamma}(t) \\ F(0,s) \equiv p_0 \\ F(1,s) \equiv p_0 \end{cases}$

(Homotopy)

•••) The identity of the group is the class of loops contractible to p_0.

π_1 essentials

For a closed, compact R.S. of genus g the group is generated by $2g$ generators $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$ subject to a single fundamental relation:

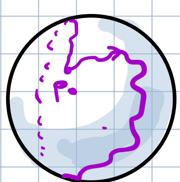
$$\boxed{\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = id}$$

Any element of the group is expressible as a noncommutative word in these $2g$ letters (and inverses)

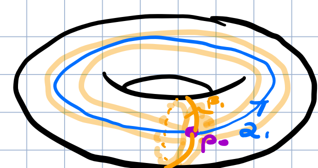
Example

Sphere (genus 0)

every closed loop is contractible $\Rightarrow \pi_1(\mathbb{P}_1, p_0) = \{id\}$



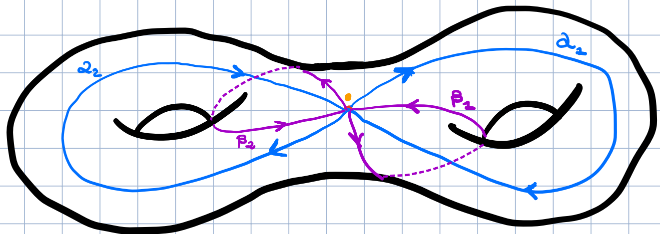
Torus (genus 1)



$\alpha\beta\alpha^{-1}\beta^{-1} = id \iff \alpha\beta = \beta\alpha$

It is a commutative group on two generators

Bitorus $(g=2)$ (and higher)



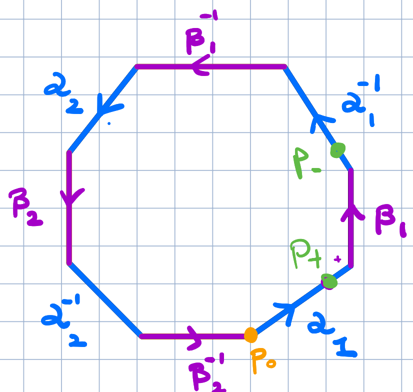
Following the generators and keeping them on the left, we come back to the start point without intersecting them.

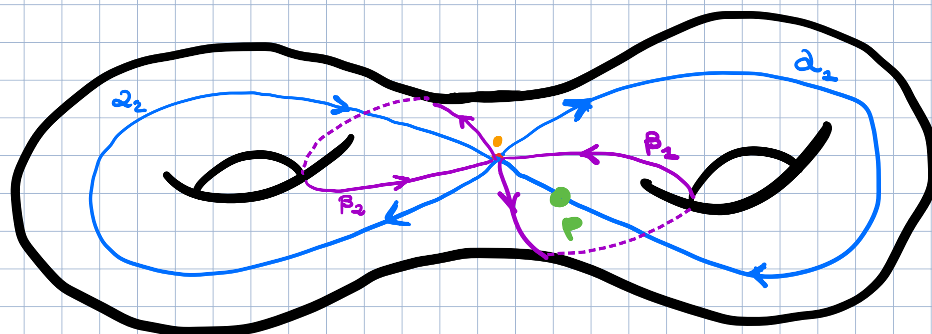
(The group is not commutative)

We trace the boundary of the canonical dissection (along the chosen generators)

Lesson: given a canonical set of generators, the dissection of $\mathcal{C}$ along their representatives gives a simply connected domain called fundamental polygon.

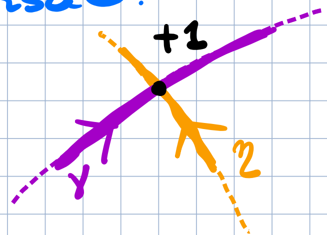
It is a domain bounded by $\boxed{4g}$ sides (each of the $2g$ generators is traversed twice in opposite directions)





Intersection number

Given γ, η we can choose representatives such that all intersections are transversal:



The intersection number $\gamma \bullet \eta$ ($\gamma \# \eta$) is the count of all intersection points, with +1 if the tangents form +vely oriented frame, −1 otherwise.

Property: i) $\gamma \bullet \eta = - \eta \bullet \gamma \quad (\in \mathbb{Z})$

ii) $\gamma \bullet \eta$ does not depend on the choice of representatives

iii) $\gamma \bullet (\eta \odot \rho) = \gamma \bullet \eta + \gamma \bullet \rho$

concatenation

Homology group (first homology group...)

It is the formalization of "contour deformation" in all computations involving Cauchy's theorem.

Lazy def $H_1(\mathbb{C},\mathbb{Z}) \simeq \dfrac{\pi_1(\mathbb{C},p_0)}{[\pi_1(\mathbb{C},p_0),\pi_1(\mathbb{C},p_0)]}$ (Abelianization of π_1)

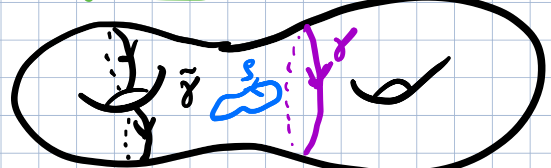
Semi-formal definition

Def A *multi-curve* is a union of closed, oriented curves

Def Two multi curves $\gamma, \tilde{\gamma}$ are **homologous** if there is a subregion $\mathcal{D} \subset \mathbb{C}$ such that the boundary consist of $\gamma - \tilde{\gamma}$ or $\tilde{\gamma}-\gamma$ (where "$-$" means the multicurve γ with the opposite orientation. This is an equivalence relation (exercise!)

Def The first homology group $H_1(\mathbb{C},\mathbb{Z})$ is the free abelian group spanned (over $\mathbb{Z}$) by homology classes of closed curves.

Examples



$\tilde{\gamma}$ is the sum of two simple loops and is homologous to γ Note that γ is also trivial.

β bounds a region $\Rightarrow$ homologically trivial.

Facts ① The intersection number is also well-defined as a bilinear, skew-symmetric pairing in $H_1(C,\mathbb{Z})$

② It is nondegenerate i.e. if $\gamma \cdot \eta = 0$ $\forall \eta \in H_1(C,\mathbb{Z})$ then $\gamma = 0$ (homologous to the null contour)

if C is w.o. boundary & compact

③ The dimension (rank) of $H_1(C,\mathbb{Z})$ is $\boxed{2g}$ (as for π_1)

④ We can always choose a canonical symplectic basis (∞'ly many, in fact)
$\mathcal{B} = \{a_1, \dots, a_g, b_1, \dots, b_g\}$ such that

$$a_i \cdot a_j = b_i \cdot b_j = 0$$

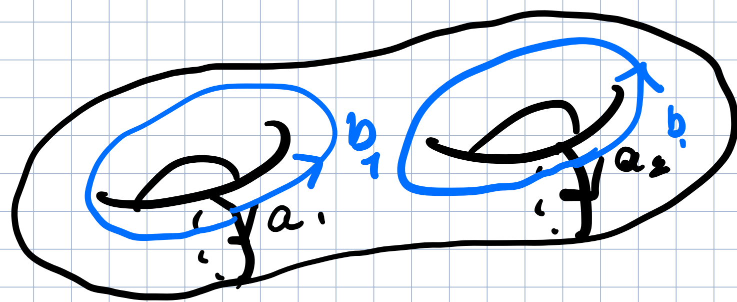
$$a_i \cdot b_j = \delta_{ij} = -b_j \cdot a_i$$

$$\begin{array}{c|cc} & a_1 \dots a_g & b_1 \dots b_g \\ \hline a_1 \dots a_g & 0_{g\times g} & \mathbb{1}_{g\times g} \\ b_1 \dots b_g & -\mathbb{1}_{g\times g} & 0_{g\times g} \end{array} =: J$$

One can choose infinitely many symplectic bases.

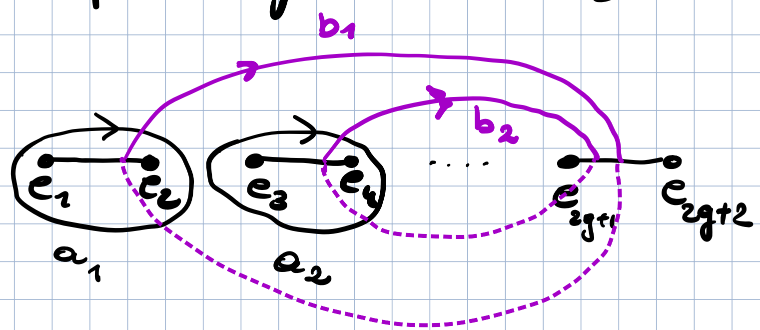
Def A Torelli marking is the choice of a symplectic basis in $H_1(C,\mathbb{Z})$

Typical picture:



Example (recurring example)

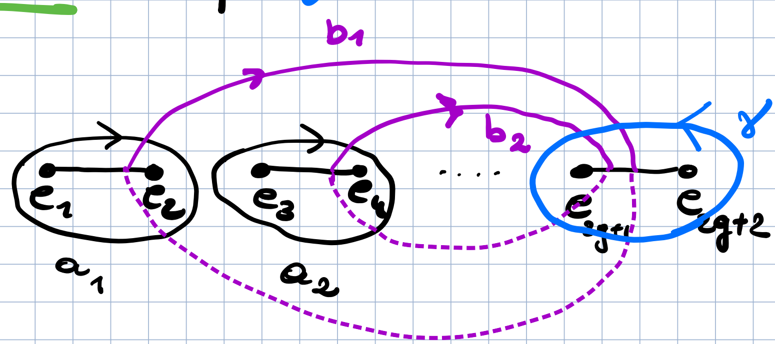
Hyperelliptic surface $\mathcal{C} = \{ y^2 = \prod_{j=1}^{2g+2} (z - e_j) \} \cup \{\infty_+, \infty_-\}$



Formula: For any contour $\gamma \in H_1(\mathcal{C}, \mathbb{Z})$ we can decompose it on a given basis

$$\gamma = \sum_{j=1}^{g} m_j a_j + n_j b_j \qquad \begin{cases} m_j = \gamma \cdot b_j \\ n_j = -\gamma \cdot a_j \end{cases}$$

Exercise Decompose γ in the above basis



Differential & Integral calculus

In 2-dim, in local real coordinates x, y a differential form with **complex values** is

$$\boxed{\omega = A(x,y)\,dx + B(x,y)\,dy} \qquad z = x+iy$$

Rewrite: $\omega = f(z,\bar z)\,dz + g(z,\bar z)\,d\bar z$ with $\begin{cases} f = \frac{A}{2} - \frac{iB}{2} \\ g = \frac{A}{2} + i\frac{B}{2} \end{cases}$

Under change of holomorphic coordinate $w = w(z)$ we have

$$\boxed{\omega = \underbrace{\tilde f\, dw}_{(1,0)\text{ part}} + \underbrace{\tilde g\, d\bar w}_{(0,1)\text{ part}}} \quad \text{with } \tilde f = f \cdot \frac{dz}{dw}\ ;\ \tilde g = g \cdot \overline{\left(\frac{dz}{dw}\right)}$$

Dolbeault decomposition

Def A form ω is **closed** if $d\omega = 0$ (exterior derivative)

namely $\boxed{\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}}$ or equivalently $\boxed{\frac{\partial}{\partial \bar z} f = \frac{\partial g}{\partial z}}$ where

$\partial, \bar\partial$ are the Wirtinger operators $\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$ $\quad \frac{\partial}{\partial \bar z} = \frac{1}{2}(\partial_x + i\partial_y)$

Remark If ω is of type $(1,0)$, i.e. $\omega = f\,dz$ then ω is closed $\Leftrightarrow$ $\frac{\partial f}{\partial \bar z} = 0$

(i.e. f is holomorphic, in any local coord.)

ω of type $(0,1)$ is closed $\Leftrightarrow$ g is antiholomorphic

Def A form ω is <u>exact</u> if there is a globally defined (smooth) function $F: \mathbb{C} \mapsto \mathbb{C}$ such that $\omega = dF = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \bar{z}} d\bar{z}$.

Remark Exact $\Rightarrow$ Closed Closed $\not\Rightarrow$ Exact

(Pseudo) Example: $S^1 = \{ [0, 2\pi] \text{ with periodic b.c.} \}$

a differential $\omega = f(\vartheta) d\vartheta$ is always closed, but it is exact iff $\int_0^{2\pi} f(\vartheta) d\vartheta = 0$

Exercise: On a closed (compact) R.S. there are no nontrivial exact holomorphic differentials.

(Proof: ω holomorphic + exact $\Rightarrow$ $\omega = df$ and f is holomorphic global function. Since $\mathrm{Re} f$ is harmonic, by the max principle it cannot have local max or min $\Rightarrow$ constant! $\Rightarrow$ $\omega = df = 0$)

Def A two-form $\eta = f(z, \bar{z}) dz \wedge d\bar{z}$ s.t. under change of coordinate

$$\eta = \tilde{f} dw \wedge d\bar{w} \quad \text{with} \quad \tilde{f} = f \cdot \left| \frac{dz}{dw} \right|^2$$

Def The first De Rham cohomology group is the (vector space) of $Z^1 = \{ \text{vect. space of smooth } \underline{\text{closed differentials}} \}$ quotiented by the subspace of <u>exact differentials</u>

End I^{st} Lecture