$\Omega^0 = \{$ space of smooth, complex-valued functions $\}$ $\qquad d\Omega^0$ alternative notation. $= \{\text{exact}\}$

$$H^1(\mathbb{C}, \mathbb{C}) = Z^1 / d\Omega^0 = \frac{\{\text{closed diffs}\}}{\{\text{exact. diffs}\}} = H^1_{dR}$$

Remark In higher dimension one considers $H^k(M, \mathbb{C}) = Z^k / d\Omega^{k-1}$ with

$\Omega^{k-1} = \{$ v. space of smooth $(k-1)$-forms $\}$

Integration: spaces in duality

Mantra: we can integrate a one form (differential) along a curve. The result is independent of the coordinate.

- If ω is **closed** the result is "independent of the path".

More precisely

Proposition If $\omega \in Z^1$ (closed one form) and γ is a (multi)contour in homology, then

$\int_\gamma \omega$ is independent of the homology class.

I.e. if $\gamma \sim \tilde{\gamma} \Rightarrow \oint_\gamma \omega = \oint_{\tilde{\gamma}} \omega$.

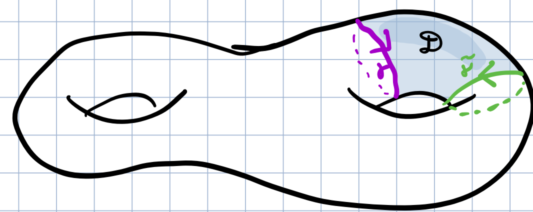
Thus: $\int_\bullet \omega : H_1(\mathbb{C}, \mathbb{Z}) \to \mathbb{C}$, $\gamma \mapsto \int_\gamma \omega$ is well defined

Proof:

•) $\gamma \sim \tilde{\gamma} \overset{def}{\Longleftrightarrow} \partial \mathcal{D} = \gamma - \tilde{\gamma}$ (or viceversa)

•) Stokes \ Green integration formula: $\displaystyle\iint_{\mathcal{D}} d\omega = \int_{\partial\mathcal{D}} \omega = \oint_{\gamma} \omega - \oint_{\tilde{\gamma}} \omega$.

Since $d\omega \equiv 0 \longrightarrow$ Q.E.D.



Fundamental corollary (Poincaré duality)

The pairing $\displaystyle\int : H_1(\mathcal{C},\mathbb{Z}) \times H^1_{dR}(\mathcal{C},\mathbb{C})$ is well defined & nondegenerate.

Proof If $\tilde{\omega} = \omega + df$ for a smooth function f then

$$\oint_{\gamma} \tilde{\omega} = \oint_{\gamma} \omega + \int_{\gamma} df = 0$$

by the fundamental thm. of calc since γ is composed of closed loops.

Nondegeneracy: If $\oint_{\gamma} \omega = 0$ for all closed loops $\Rightarrow \omega$ is exact,

namely, $\omega = 0$ in H^1_{dR} $\blacksquare$ $\quad \omega = df \quad , \; f = \int_{P_0}^{P} \omega$

The Riemann Bilinear Identity

(For any cycle $\gamma \in H_1(C,\mathbb{Z})$ and $\omega \in H^1_{dR}$, we call $\oint_\gamma \omega$ the period of ω along γ)

Preparation: given two closed forms $\omega = f\,dz + g\,d\bar z$, $\eta = h\,dz + k\,d\bar z$ $\left(\frac{\partial f}{\partial \bar z} = \frac{\partial g}{\partial z} \text{ etc.}\right)$

their wedge product $\omega \wedge \eta = [f \cdot k - g \cdot h]\, dz \wedge d\bar z = (\star)\, dx \wedge dy$

is a "volume (area) form" (2-form) and we can integrate on the surface.

Theorem Let $a_1 \cdots a_g, \beta_1 \cdots \beta_g$ be canonical contours in $\pi_1(C, p_0)$ so that
$a_1 \beta_1 a_1^{-1} \beta_1^{-1} \cdot \cdots \cdot a_g \beta_g a_g^{-1} \beta_g^{-1} = id.$

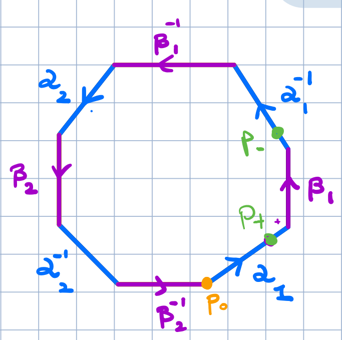
Let $\omega, \eta \in H^1_{dR}$ (or some representative)

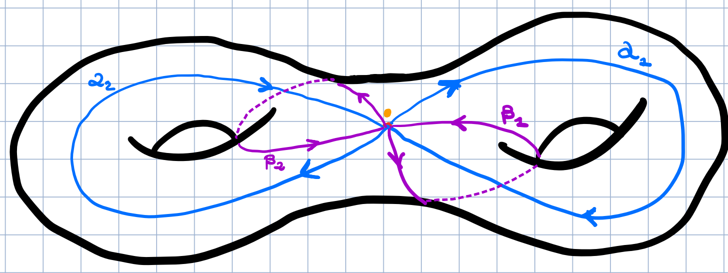
Let $\mathcal{L}$ denote the canonical dissection (simply connected) of C along them.

Then: (Riemann Bilinear Identity)

$$\iint_C \omega \wedge \eta = \sum_{j=1}^{g} \oint_{a_j} \omega \oint_{\beta_j} \eta - \oint_{a_j} \eta \oint_{\beta_j} \omega$$

Proof





$$\iint_{\mathcal{C}} \omega\wedge\eta = \iint_{\mathcal{L}} \omega\wedge\eta .$$

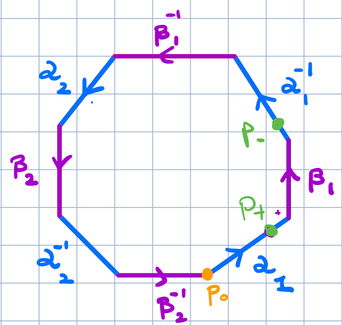
contours are of zero Leb. measure

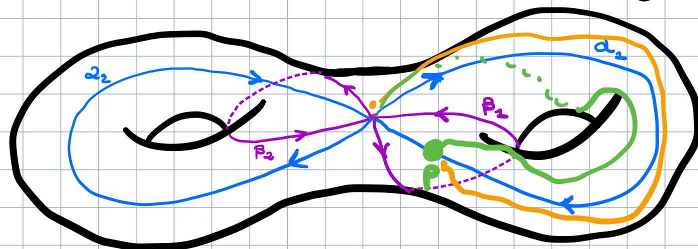
$\mathcal{L}$ is simply connected so $\omega = dF$ for some smooth function

$$F(p) = \int_{p_0}^{p} \omega$$ along a path that does not cross any α, β.

(example on figure!)

If $p \in \alpha_j$, we can reach it from the left (p_+) or from the right (p_-)





$$F(p_+) = F(p_-) + \int_{p_-}^{p_+} \omega = F(p_-) - \oint_{\beta_j} \omega$$

Similarly (note opposite sign!) $p \in \beta_j$ $\qquad F(p_+) = F(p_-) + \oint_{\alpha_j} \omega$

Now: $\omega \wedge \eta = d(F \cdot \eta)$ is exact on $\mathcal{L}$ (not $\mathcal{C}$!)

Hence by Stokes\Green theorem

$$\iint_{\mathcal{L}} \omega\wedge\eta = \oint_{\partial\mathcal{L}} F\eta = \int_{\alpha_1} F(p_+)\cdot\eta(p) + \int_{\beta_1} F(p_+)\,\eta(p) - \int_{\alpha_1} F(p_-)\,\eta(p) - \int_{\beta_1} F(p_-)\,\eta(p) + \dots =$$

$$= \int_{\alpha_1} \overbrace{[F(p_+)-F(p_-)]}^{-\oint_{\beta_1}\omega}\,\eta(p) + \int_{\beta_1} \underbrace{[F(p_+)-F(p_-)]}_{\oint_{\alpha_1}\omega}\,\eta(p) + \dots = \oint_{\alpha_1}\omega \cdot \oint_{\beta_1}\eta - \oint_{\alpha_1}\eta \oint_{\beta_1}\omega + \dots \blacksquare$$

Bingo!!

Consequences

Let $\omega = f(z)\,dz$ be a <u>holomorphic</u> (hence closed) differential

We denote $\overline{\omega} = \overline{f(z)}\,d\bar{z}$ (antiholomorphic, also closed).

Apply RBI to $\omega = \omega$, $\eta = \overline{\omega}$ (set $A_j = \oint_{\alpha_j} \omega$; $B_j = \oint_{\beta_j} \omega$)

$$\iint \omega \wedge \overline{\omega} = \sum_{j=1}^{g} \oint_{\alpha_j} \omega \oint_{\beta_j} \overline{\omega} - \oint_{\alpha_j} \overline{\omega} \oint_{\beta_j} \omega = 2i\, \Im m \left[\sum_{j=1}^{g} A_j \overline{B}_j \right]$$

On the other hand: $\omega \wedge \overline{\omega} = |f(z)|^2 dz \wedge d\bar{z} = -2i\, |f(z)|^2 dx \wedge dy$

Therefore:

$$\Im m \left[\sum_{j=1}^{g} A_j \overline{B}_j \right] \leq 0 \quad \text{for any holomorphi } \omega$$

The equality can hold iff $|f| \equiv 0$ (in all coordinate charts)

<u>Corollary</u> •) All α_j periods (or all β_j periods) of a holomorphic differential are zero iff $\omega = 0$

••) All periods (α & β) are real (imaginary) iff $\omega = 0$

{Mero, Holo}morphic differentials

Facts (i.e. theorems)

Theorem

① For any R.S. of genus g there are g, linearly independent holomorphic differentials.

Important given any Torelli marking $\{a_1, \dots, a_g, b_1 \dots b_g\}$ and any basis of holomorphic differentials $\{\eta_1, \dots, \eta_g\}$ there is a normalized basis $\{\omega_1, \dots \omega_g\}$ such that:

$$\oint_{a_i} \omega_j = \delta_{ij} \quad , i,j = 1 \dots g$$

Exercise: The matrix $A_{je} = \oint_{a_j} \eta_e$ is <u>invertible</u> (why? ... RBI). Then define $\omega_j = \sum_k (A^{-1})_{kj} \eta_k$ ....

② Any meromorphic differential η can be "a-normalized" (uniquely) by adding a lin.combo. of ω_j's so that

$$\oint_{a_j} \eta = 0 \qquad \forall j = 1 \dots g$$

③ Given any pair of points q_+, q_- there is a 3 kind diff

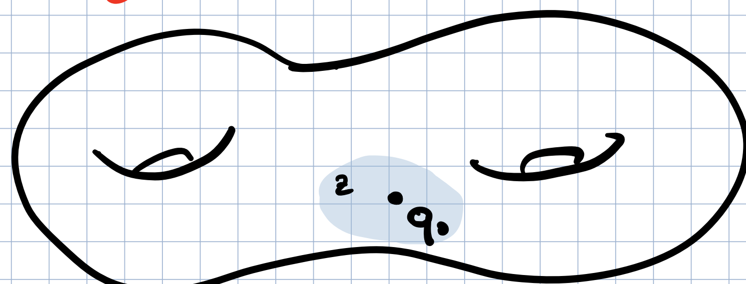
$\Omega_{q_+,q_-}(p)$ such that $\boxed{\operatorname{res}_{p=q_\pm} \Omega_{q_+,q_-} = \pm 1}$

④ For any p_0, $k=1,2\ldots$ and local coord. z such that $z(p_0)=0$ there is a 2^{nd} kind differential $\Omega_k(p)$ such that it has only a pole at p_0 of order $k+1$ and expansion of the form:

$$\Omega_k(p) = \left[\frac{1}{z^{k+1}} + O(1)\right] dz \quad \circledast \quad (z = z(p),\ p \to p_0)$$

The fundamental bidifferential ("Bergman")

Take $\Omega_2(p)$ with pole at q_0 and a-normalized.



$$\Omega_2(p) = \left[\frac{1}{(z(p)-z(q_0))^2} + O(1)\right] dz(p)$$

$$\oint_{a_j} \Omega_2 = 0$$

The result <u>does</u> depend on choice of local coord, if $z \to w$

Exercise

$$\boxed{\tilde{\Omega}_2(p) = \frac{dz(q_0)}{dw(q_0)} \Omega_2(p)}$$

This suggests: promote the q_0-dependence to differential!

Def $B(p;q)$ is the (unique) bi-differential (i.e. differential w.r.t. both p,q) such that

① $\mathrm{div}_p\big(B(p;q)\big) \geq -2(q)$ (has a double pole at $p=q$)

② $$B(p;q) = \left[\frac{1}{(z(p)-z(q))^2} + \frac{1}{6} S_B(q) + \mathcal{O}\big(z(p)-z(q)\big)\right] dz(p)\, dz(q)$$

③ $$\oint_{p\in a_j} B(p;q) = 0 \quad \forall j = 1\dots g \quad (a\text{-normalized})$$

④ $B(p;q) = B(q;p)$ (symmetry)

Properties ① $$\oint_{p\in b_j} B(p;q) = 2\pi i\, \omega_j(q) \quad (\leftarrow \text{RBI})$$

Exercise

② The regular term in the diagonal expansion is the "Bergman projective connection"

Under change of coord.

$$\Omega(p,q) = \left[\frac{1}{(z-w)^2} + \frac{1}{6} S_B(w) + O(z-w) \right] dz\, dw$$

$z = \zeta(p) \quad w = \zeta(q)$

ζ a local coord.

Bergman projective (affine) connection (i.e. stress-energy tensor in c=1 CFT)

$$S_B(\tilde{w}) \left(\frac{d\tilde{w}}{dw} \right)^2 = S_B(w) + \{\tilde{w}, w\}$$

Schwarzian derivative

$$\{\tilde{w}, w\} = \left(\frac{\tilde{w}''}{\tilde{w}'} \right)' - \frac{1}{2} \left(\frac{\tilde{w}''}{\tilde{w}'} \right)^2 \qquad \left(\tilde{w}' = \frac{d\tilde{w}}{dw} \right)$$

(3) Any other $\Omega_k(p)$ is obtained from B as follows:

choose q_0 and pointed local coordinate $\zeta : \mathcal{U} \to \mathbb{C}$ $\quad (\zeta(q_0) = 0)$

Then

$$\Omega_k(p) = \operatorname*{res}_{q = q_0} \frac{(-k)}{\zeta(q)^k} B(p;q)$$

Exercise: In local coord ζ, $\Omega_k(p) = f(\zeta)\, d\zeta = \operatorname*{res}_{\zeta' = 0} \frac{(-k)}{\zeta'^k} \cdot \left[\frac{1}{(\zeta - \zeta')^2} + \cdots \right] d\zeta' d\zeta$

Use Cauchy's residue formula to see the singular behavior: check a-periods.

There is an explicit formula for B in terms of Θ-functions
(later)

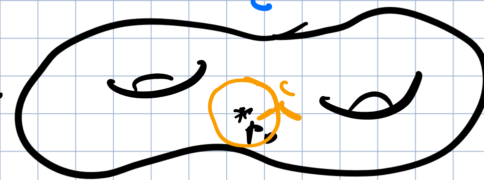
Mero/holomorphic differentials

Def Similar to the case of functions

$$\boxed{\operatorname{div}(\omega) = \sum_{p\in C} \operatorname{ord}_\omega(p)\,(p)}$$

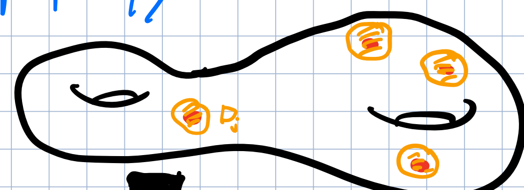
Given a pole p_0 of ω and a small loop $\qquad \operatorname{res}_{p=c}\omega \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_c \omega$

The value of residue does not depend on the choice of coordinate used to compute it.



Prop: If η is any meromorphic differential $\displaystyle\sum_{p=\text{pole of }\eta} \operatorname{res}_p \eta = 0$

Proof By Stokes/Green



$$0 = \iint_{C/\text{disks}} \overset{\overset{0}{\shortparallel}}{\overline{d\eta}} = \sum \oint_{\partial D_j} \eta = 2\pi i \sum_{\text{poles}} \operatorname{res}_p \eta$$

$\blacksquare$

Normalized holomorphic differentials

Let C be of genus g, $\{\alpha_1 \dots \alpha_g, \beta_1 \dots \beta_g\}$ a Torelli marking

Def The normalized basis of holomorphic differentials $\omega_1, \dots \omega_g$ such that

$$\boxed{\oint_{\alpha_j} \omega_k = \delta_{jk}}$$

Nota bene For any basis $\eta_1 \dots \eta_g$ the matrix $A_{ij} := \oint_{\alpha_i} \eta_j$ is invertible (Exercise). Then $\begin{pmatrix}\omega_1 \\ \vdots \\ \omega_g\end{pmatrix} = A^{-1}\begin{bmatrix}\eta_1 \\ \vdots \\ \eta_g\end{bmatrix}$

Theorem (Riemann) Define the matrix of normalized β period as $\Pi_{ij} := \oint_{\beta_i} \omega_j$.

Then ① $\Pi_{ij} = \Pi_{ji}$;

② $\mathrm{Im}(\Pi) > 0$ (positive definite)

Proof. All consequence of RBI applied to ω_i, ω_j.

① Since $\omega_i \wedge \omega_j \equiv 0$ (why?... ($dz \wedge dz = 0$))

$$0 = \iint_C \omega_i \wedge \omega_j = \sum_{\ell=1}^{g} \underbrace{\oint_{\alpha_\ell} \omega_i}_{=\delta_{i\ell}} \oint_{\beta_\ell} \omega_j - \underbrace{\oint_{\alpha_\ell} \omega_j}_{=\delta_{j\ell}} \oint_{\beta_\ell} \omega_i = \Pi_{ij} - \Pi_{ji}$$

② Let $\omega = \sum c_j \omega_j$, $c_j \in \mathbb{C}$.

Then
$$0 \geq \frac{1}{2i} \iint_{\mathcal{L}} \omega \wedge \bar{\omega} = \frac{1}{2i} \sum_{l,j,k} c_j \overset{\delta_{lj}}{\oint_{\alpha_l} \omega_j} \oint_{\beta_l} \bar{\omega}_k \bar{c}_k - \bar{c}_k \overset{\delta_{lk}}{\oint_{\alpha_l} \bar{\omega}_k} \oint_{\beta_l} \omega_j c_j =$$
$$= \frac{1}{2i}\left[c^{t} \cdot \overline{\Pi} \cdot c - c^{t} \cdot \Pi c \right] = - c^{t} \cdot (\mathrm{Im}\, \Pi) \cdot c \qquad \blacksquare$$

Maybe?

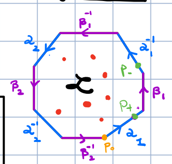
Reciprocity Theorems (relate β-period to residues)

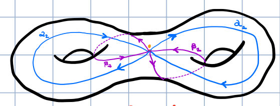
Let η, ω be meromorphic differentials, with at least one (ω) of second kind (i.e. no residues)

Let $\mathcal{L}$ be the canonical dissection. Let $F = \int_{p_0}^{p} \omega$; it is a single-valued meromorphic function on $\mathcal{L}$.

Now integrate $F \cdot \eta$ along $\partial \mathcal{L}$ and use Cauchy's theorem:

$$\boxed{\sum_{l=1}^{g} \oint_{\alpha_l} \omega \oint_{\beta_l} \eta - \oint_{\alpha_l} \eta \oint_{\beta_l} \omega = 2\pi i \cdot \sum_{q \in \text{poles of } F \cdot \eta} \operatorname{res}_{q} F \cdot \eta}$$





. poles of F and/or η

Special cases of interest

‖ A meromorphic differential η on a Torelli marked R.S. is called α-normalized if
$$\oint_{\alpha_j} \eta = 0 \quad \forall j = 1 \dots g.$$

Suppose η has a single pole at p_0 of order $k+1$ with singular part
$$\eta = -k\left(\frac{1}{z^{k+1}} + O(1)\right) dz \qquad (k = 1, 2, \dots)$$
convenience normalization

Special case I
$$\oint_{\beta_j} \eta = \operatorname{res}_{p_0} \frac{1}{z^k(p)} \omega_j(p)$$

Proof:
$$\sum_{l=1}^{g} \oint_{\alpha_l} \omega_j \oint_{\beta_l} \eta - \underbrace{\oint_{\alpha_l} \eta}_{=0 \text{ because of normalization}} \oint_{\beta_l} \omega_j = \oint_{\beta_j} \eta = 2\pi i \operatorname{res}_{p_0} \Big(\overbrace{\int \eta}^{\left[\frac{1}{z^k} + O(1)\right]} \Big) \cdot \omega_j$$

Special case II $\Omega_{p_+ p_-}$ the α-normalized 3^{rd} kind differential
$$\oint_{\beta_j} \Omega_{p_+ p_-} = 2\pi i \int_{p_-}^{p_+} \omega_j \qquad \text{(path not crossing the marking } \alpha, \beta \text{'s)}$$

Some concrete example : Plane curves

A plane curve is the Locus in $\mathbb{C}^2$

$$F(z,w) = \sum a_{ij} z^i w^j = 0$$

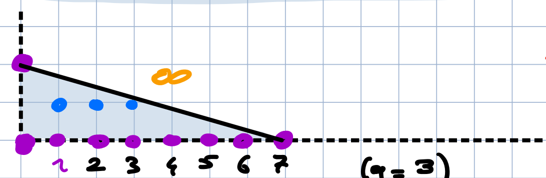
Assume: non singular, i.e. $F_z = F_w = F = 0$ has no solutions

Newton's polygon: $\mathcal{N} = \text{Convex Hull}\left\{ (i,j) \in \mathbb{Z}^2 : a_{ij} \neq 0 \right\}$

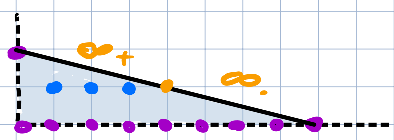
From this we can read off
- •) # of pts at $z=\infty$ ($w=\infty$) together with local coord.
- ••) # of holomorphic diffs.

$z^{\frac{1}{\text{drop}}}$

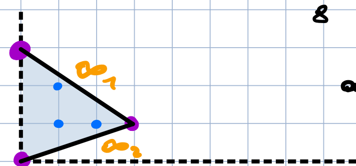
•) $w^2 = P_{2g+1}(z)$



•) $w^2 = P_{2g+2}(z)$



•) $w^3 + z^3 w + z = 0$



at ∞_1 $w = \pm z^{3/2}$ (Puiseux in $z^{1/2}$)

∞_2 $w = \frac{1}{z^3}$ (Taylor in z^{-1})

$(g=3)$

$w^3 + z^3 + z^3 w + z = 0$

(The # of pts at ∞ is # of sides facing right; Puiseux series in $z^{1/k}$ where k is the drop of the side)

Rule: The differentials $\omega_{(i,j)} := \frac{z^{i-1} w^{j-1} dz}{F_w} = \frac{-z^{i-1} w^{j-1} dw}{F_z}$

are holomorphic $\forall$ (i,j) in the interior of N

Nota bene: on $F(z,w)=0$ we have $F_z dz = -F_w dw$, hence this

The next 2 slides are FYI

about the Riemann–Roch theorem: will not

be covered in class. An important consequence is

Fact: ① if f is a meromorphic function then $\deg(\mathrm{div}(f)) = 0$

② if ω is any meromorphic (or holomorphic) differential, then

$$\deg \mathrm{div}(\omega) = 2g-2$$

Exercise For $y^2 = P_{2g+2}(z)$ (compactified), verify for $\eta = \frac{z^\ell dz}{y}$

$(\ell \leq g-1)$ holomorphic differential

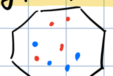
(FYI but skip in class)

Riemann-Roch

The divisor of a meromorphic function is called principal

Prop Every principal divisor has zero degree.

Proof



• poles
• zeros

(Assume a, β's are chosen/deformed to avoid zeros/poles)

By the argument principle $\oint_{\partial\Sigma} \frac{df}{f} = \{\text{total \# zeros}\} - \{\text{total \# poles}\} = \deg(\operatorname{div}(f))$

OTOH $\oint_{\partial\Sigma} \frac{df}{f} = 0$ because the integral traverses each a, β's twice in opposite directions and $\frac{df}{f}$ takes same values on +/- bdry vals ∎

Question: what about the converse?

Ex. $\boxed{g=0}$ $\mathcal{D} = (z_1) + (z_2) + \cdots + (z_N) - (p_1) - \cdots - (p_N)$

arbitrary divisor of degree 0. (Pts possibly repeated: for simplicity all $\neq \infty$)

Then $$f(z) = * \frac{\prod_{j=1}^{N}(z - z_j)}{\prod_{j=1}^{N}(z - p_j)}$$ does the trick.

$\boxed{\text{In genus } g \geq 1}$ We'll see that not all divisors of degree 0 are principal.

The pts of the divisor must satisfy g extra conditions.

Def (Partial order) Given two divisors $\mathcal{D} = \sum_{p \in \mathcal{C}} k_p \cdot (p)$, $\widetilde{\mathcal{D}} = \sum_{p \in \mathcal{C}} \tilde{k}_p \cdot (p)$

we say $\mathcal{D} \geq \widetilde{\mathcal{D}}$ if $k_p \geq \tilde{k}_p$ $\forall p$.

Some more consequences

Let ω be any meromorphic differential: what is $\deg \operatorname{div}(\omega)$?

First: $\operatorname{div}(\omega)$ is a divisor class (modulo linear equivalence) independent of ω. Indeed,

if ω, η are $\{\text{mero/holo}\}$morphic differentials, then $f(z) = \frac{\omega}{\eta}$ is a meromorphic function.

(proof: $\omega = h(z)dz$, $\eta = g(z)dz \Rightarrow f(z) = \frac{h(z)}{g(z)}$
$= \tilde{h}(w)dw$, $= \tilde{g}(w)dw$

$\tilde{h} = h \cdot \frac{dz}{dw}$ $\tilde{g} = g \cdot \frac{dz}{dw} \rightarrow f(w) = \frac{\tilde{h}(w)}{\tilde{g}(w)} = \frac{h(z(w))}{g(z(w))}$)

We call this class the canonical (divisor) class $\mathcal{K}$.

Then $\boxed{\textbf{Prop. } \deg K = 2g - 2}$

(If $\mathcal{D} = \operatorname{div}(\omega)$, ω holomorphic, then

$r(-\mathcal{D}) = g$ and $\mathcal{R}(\mathcal{D}) = \mathbb{C}\{1, \frac{\omega_2}{\omega}, \ldots, \frac{\omega_g}{\omega}\}$ holo

so $g = \underset{1}{i(\mathcal{D})} + \deg \mathcal{D} - g + 1 = \deg \mathcal{K} - g + 2$

Def Let $\ell = 1, 2, \ldots$ A ℓ-differential is $\omega = f(z)dz^\ell = \tilde{f}(w)\, dw^\ell$ with

$\tilde{f}(w) = f(z) \cdot \left(\frac{dw}{dz}\right)^\ell$ (Section of the ℓ-th power of the canonical line-bundle $\mathcal{K}^\ell$)

Def. $\mathcal{R}(\mathcal{D}) = \{f : f \text{ meromorphic, such that } \operatorname{div}(f) \geq \mathcal{D}\}$

$\mathcal{J}(\mathcal{D}) = \{\omega : \text{merom. diff. such that } \operatorname{div}(\omega) \geq \mathcal{D}\}$

$r(\mathcal{D}) = \dim \mathcal{R}(\mathcal{D})$; $i(\mathcal{D}) = \dim \mathcal{J}(\mathcal{D})$

Prop If $\mathcal{D} - \widetilde{\mathcal{D}}$ is principal ("linear equivalence") then $r(\mathcal{D}) = r(\widetilde{\mathcal{D}})$

Proof $\exists f : \operatorname{div}(f) = \mathcal{D} - \widetilde{\mathcal{D}}$. $\forall h \in \mathcal{R}(\mathcal{D})$ then $h/f \in \mathcal{R}(\widetilde{\mathcal{D}})$ and $\forall g \in \mathcal{R}(\widetilde{\mathcal{D}})$ then $g \cdot f \in \mathcal{R}(\mathcal{D})$ ∎

Examples •) $\exists f$ $\mathcal{D} = 0$ $r(\mathcal{D}) = 1$ (only constants)
$i(\mathcal{D}) = g$ (all hol. differentials)

Riemann-Roch Theorem

For any $\mathcal{D}$ $$r(-\mathcal{D}) = i(\mathcal{D}) - g + 1 + \deg \mathcal{D}$$

Note: The proof is not difficult and boils down to the "nullity + rank" theorem. See notes.

Prop The holomorphic $\ell = 2$ differential are a vector space of dimension $3g - 3$ $(g \geq 2)$

Why bother? In a more in-depth course we could study the moduli space of (smooth, genus-g, compact) Riemann Surfaces,

$$\mathcal{M}_g = \{\mathcal{C} = \text{smooth, cpct, genus}(\mathcal{C}) = g\} / \sim$$

where $\sim$ is the equivalence relation: $\mathcal{C} \sim \hat{\mathcal{C}} \Leftrightarrow \exists \varphi : \mathcal{C} \to \hat{\mathcal{C}}$ holomorphic, $\varphi^{-1} : \hat{\mathcal{C}} \to \mathcal{C}$ also holomorphic.

Fact: $\dim \mathcal{M}_g = 3g - 3$ $(g \geq 2)$ $(\dim \mathcal{M}_{1,1} = 1)$

Answer: $H_0(\mathcal{K}_c^2) = \{\text{vect. space of holomorphic quadratic differentials}\}$ is isomorphic (Beltrami) to the co-tangent space of $\mathcal{M}_g$ at $[\mathcal{C}] \in \mathcal{M}_g$

Consequences

If $\mathcal{D}$ is a positive divisors, $\mathfrak{I}(\mathcal{D})$ is a subspace of holomorphic differentials and hence $i(\mathcal{D}) \leq g$.

In general, if $\mathcal{D} = (p_1) + \dots + (p_k)$, $\mathfrak{I}(\mathcal{D})$ consists of

hol. diffs. that vanish at $p \in \{p_1 \dots p_k\}$ so that (generically)

there are k linearly independent constraints. Thus

$$i(\mathcal{D}) = g - k \quad \text{and}$$

$$r(-\mathcal{D}) = g - k - g + 1 + k = 1 \qquad \text{if } k \leq g$$

(unless $\mathcal{D}$ is "special", i.e. the constraints are dependent)

For example: if $\mathcal{D} = (p_1) + \dots + (p_g)$ (distinct points for simplicity)

then the $i(\mathcal{D})$ is the corank of the matrix

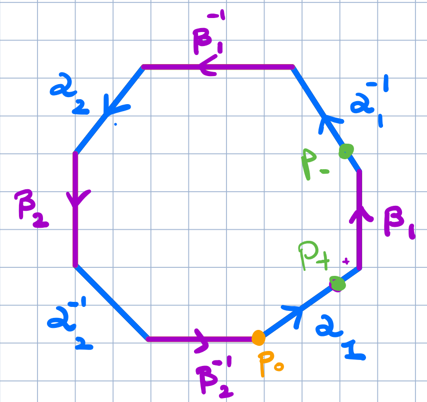
$$\Delta(\mathcal{D}) = \begin{bmatrix} \omega_1(p_1) & \cdots & \omega_g(p_1) \\ \omega_1(p_2) & \cdots & \omega_g(p_2) \\ & & \\ \omega_1(p_g) & \cdots & \omega_g(p_g) \end{bmatrix}$$

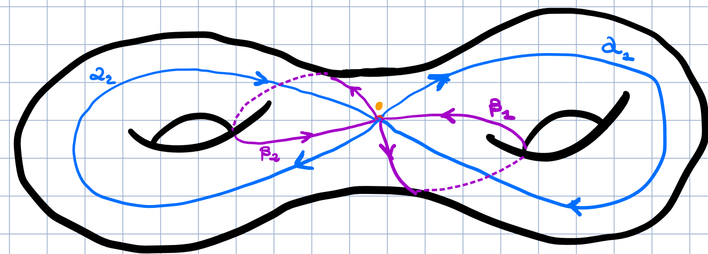
(evaluation of ω's in any local coord. does not affect the rank) Exercise!

Generically $\det \Delta \neq 0$ and hence $\mathfrak{I}(\mathcal{D}) = \{0\}$ $(i(\mathcal{D}) = 0)$

If $\mathcal{D}$ is special (of degree g) then $i(\mathcal{D}) > 0$

Abel map





Fix:
- Torelli marking $\{\alpha_1 .. \alpha_g, \beta_1 .. \beta_g\}$
- Normalized basis $\omega_1, .. \omega_g$, $\oint_{\alpha_j} \omega_i = \delta_{ij}$
- B-period matrix $\tau_{ij} = \oint_{\beta_i} \omega_j = \tau_{ji}$

Riemann Theorem

Def The **Abel map** (with basepoint p_0) is

$$\mathcal{A}(p) = \begin{bmatrix} \int_{p_0}^{p} \omega_1 \\ \vdots \\ \int_{p_0}^{p} \omega_g \end{bmatrix} \in \mathbb{C}^g$$

(Same contour of integration for all components.

Properties For brevity we denote $\mathcal{A}(p^\gamma)$ the analytic continuation along a contour γ. Here what matters is not the homotopy but the homology class of γ, so we can write $\mathcal{A}(p+\gamma)$.

•) $\mathcal{A}(p+\alpha_j) = \mathcal{A}(p) + \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \begin{array}{l} \\ \\ \to j \\ \\ \\ \end{array} \end{array} = e_j$

••) $\mathcal{A}(p+\beta_j) = \mathcal{A}(p) + \Pi \cdot e_j$

•••) In general, if $\gamma = \sum m_j \alpha_j + n_j \beta_j$ (in homology) then

$$\mathcal{A}(p+\gamma) = \mathcal{A}(p) + \vec{m} + \Pi \cdot \vec{n}, \quad \vec{m}, \vec{n} \in \mathbb{Z}^g.$$

Exercises!

Jacobian variety: $\mathbb{J} = \mathbb{J}(C) = \mathbb{C}^g / \mathbb{Z}^g + \Pi \cdot \mathbb{Z}^g$

i.e. $\mathbb{C}^g$ modulo the equiv. rel.

$$\underline{z} \sim \underline{\tilde{z}} \iff \underline{z} - \underline{\tilde{z}} = \vec{m} + \Pi \cdot \vec{n} \text{ for some } \vec{m}, \vec{n} \in \mathbb{Z}^g.$$

$\mathbb{J}(C)$ is a g-dimensional complex torus, (2g-dim. real torus)

The Abel map is well defined as a map $\mathcal{A} : C \mapsto \mathbb{J}(C)$

For any divisor $\mathcal{D} = \sum k_j (p_j)$, $k_j \in \mathbb{Z}$ we extend the definition

$$\mathcal{A}(\mathcal{D}) := \sum_j k_j \mathcal{A}(p_j) = \sum k_j \int_{p_0}^{p_j} \vec{\omega} \qquad \left(\vec{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_g \end{bmatrix} \right)$$

FYI but skip in class: Abel & Jacobi Theorems

Main Theorems: Abel & Jacobi

Theorem (Abel)

Suppose $\mathcal{D} = \mathcal{D}_+ - \mathcal{D}_-$ is a divisor of degree zero ($\mathcal{D}_+$ the positive part, $\mathcal{D}_-$ " negative part)

Then: $\mathcal{D}$ is principal (i.e. the divisor of zeros/poles of a meromorphic function)

iff $\mathcal{A}(\mathcal{D}) \equiv 0 \in \mathbb{J}(\mathcal{C})$ or, equivalently,

$\exists \vec{m}, \vec{n} \in \mathbb{Z}^g$ s.t. $\mathcal{A}(\mathcal{D}) = \vec{m} + \mathbb{T}\vec{n}$.

Remark These are the g conditions we mentioned earlier.

Remark The proof is actually constructive. see notes.

Remark Observe the contrast with the case of $\mathbb{P}^1$, where there are $g=0$ conditions (i.e. any degree-zero divisor is principal)

Remark The set of zero-degree divisors is an Abelian group. The principal divisors are a subgroup and then we can think abstractly

$$\mathbb{J}(\mathcal{C}) = \{\text{Degree zero divisors}\} / \{\text{Principal divisors}\}$$

(thanks to Abel's theorem) $\rightarrow$ "Abelian variety" (i.e. a variety with Abelian group structure)

Jacobi Theorem

●) For any $\underline{e} \in \mathbb{J}(\mathcal{C})$ there is $\mathcal{D}$, positive divisor of degree g, such that $\underline{e} = \mathcal{A}_{p_0}(\mathcal{D})$

In other words: the map from the set of +ve divisors of degree g, $\mathcal{A}(\mathcal{D}_g)$ is surjective.

●●) The map is injective on the non-special divisors $\mathcal{D}_g$ of degree g, i.e. $i(\mathcal{D}_g) = 0$

Remark on some aspects of proof.

If $\mathcal{D}_g^0 = (p_1^0) + \cdots (p_g^0)$ (Distinct for simplicity) Then the Jacobian matrix of the Abel map is

$$\mathcal{A}(\mathcal{D}_g^0) = \mathcal{A}(p_1^0) + \cdots + \mathcal{A}(p_g^0) \rightarrow d\mathcal{A}(\mathcal{D}_g^0) = \begin{bmatrix} \omega_1(p_1^0) & \cdots & \omega_1(p_g^0) \\ \omega_g(p_1^0) & & \omega_g(p_g^0) \end{bmatrix}$$

Then $\det[d\mathcal{A}] \neq 0$ exactly iff $i(\mathcal{D}_g^0) = 0$

(see comments around Riemann-Roch)

Use of Θ functions.

They are used to construct canonical objects:

- Cauchy kernels (to solve boundary value problems)
- Fundamental bidifferential (a.k.a. "Bergman kernel") wrong name
  - Chekhov–Eynard–Orantin topological recursion
- Projective connections (a.k.a. opers $\leftrightarrow$ BPS states)
- Szegö kernels ($\to$ det. of $\bar{\partial}$ operators)

Θ-functions satisfy many functional identities, almost all of which reducible to **Fay-Identities.**

Θ-functions with characteristics

(Generalization of Jacobi's elliptic ϑ_j). For $\vec{n}, \vec{m} \in \mathbb{Z}^g$ one looks at the **half-periods**:

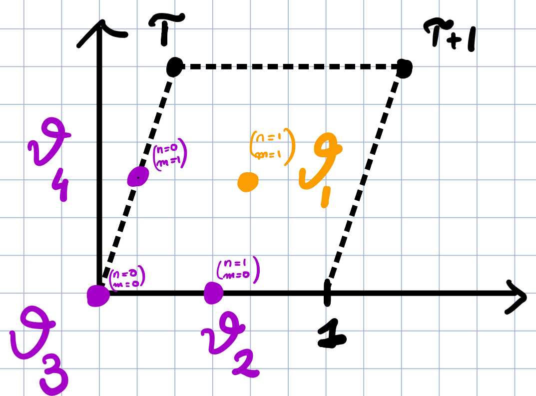
$$[n,m] = \Delta = \Delta_{\vec{n},\vec{m}} = \frac{\vec{n}}{2} + \mathbb{T} \cdot \frac{\vec{m}}{2}.$$

Def
$$\Theta_\Delta(\underline{z}) := \exp\left[2\pi i \left(\frac{1}{8}\vec{m}^t \mathbb{T} \vec{m} + \frac{1}{2}\vec{m}\cdot \underline{z} + \frac{1}{4}\vec{m}\cdot\vec{n}\right)\right] \Theta(\underline{z} + \Delta)$$

Property
$$\Theta_\Delta(-z) = e^{i\pi \vec{n}\cdot\vec{m}}\, \Theta_\Delta(z)$$

Thus we split the half periods into even/odd depending on the parity of $\vec{n}\cdot\vec{m}$.

Example (genus 1) $\quad \mathbb{T}(\tau) = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \qquad (\Im m\, \tau > 0)$



If $\Delta = \frac{a}{2} + \tau \frac{b}{2}$, $a, b \in \{0,1\}$ then

$$\Theta_\Delta(z) = \sum_{n \in \mathbb{Z} + b/2} \exp\left(i\pi n^2 \tau + 2\pi i n \left(z + \frac{a}{2}\right)\right)$$

$$\Theta_{[0,0]} = \vartheta_3 \qquad \Theta_{[1,0]} = \vartheta_2 \qquad \Theta_{[0,1]} = \vartheta_4 \qquad \Theta_{[1,1]} = \vartheta_1$$

Note: For Δ an odd half-period we necessarily have $\Theta_\Delta(\underline{0}) = 0 \Rightarrow \Delta \in \mathrm{div}(\Theta)$

(Mumford)

There exists an odd, nonsingular, half period namely $\Delta = \frac{\vec{n}}{2} + \tau \cdot \frac{\vec{m}}{2}$ $(\vec{n}, \vec{m} \in \{0,1\}^g)$, $\vec{n} \cdot \vec{m} \in 2\mathbb{Z}+1$ (odd) such that:

$$\text{(gradient!)} \to \nabla \Theta(\Delta) \neq \underline{0}$$

Foy identities

Let $p_1, \dots p_N$, $q_1 \dots q_N \in C$, Let $\underline{e} \in J(C) \setminus \mathrm{div}(\Theta)$

<u>Then</u>

$$\det\left[\frac{\Theta(A(p_i) - A(q_j) + \underline{e})}{\Theta(e)\,\Theta_\Delta(A(p_i) - A(q_j))}\right]_{i,j=1}^{N} = \frac{\prod_{i<j} \Theta_\Delta(A(p_i) - A(p_j))\,\Theta_\Delta(A(q_i) - A(q_j))}{\prod_{i,j=1}^{N} \Theta_\Delta(A(p_i) - A(q_j))} \cdot \frac{\Theta\left(A\left(\sum_i^N (p_i - q_i)\right) + e\right)}{\Theta(e)}$$

(For $N=3 \to$ Foy trisecant id.)

Remark

This is a higher genus generalization of Cauchy's determinantal identity

$$\det\left[\frac{1}{x_i - y_j}\right] = \frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(x_i - y_j)}$$

Examples/exercises.

In genus $g=1$: There is only **one** odd characteristic $[1,1] \to$ Jacobi $\vartheta_1(z)$

Then Fay identities read: $K(v,s) := \dfrac{\vartheta_3(v-s+e)}{\vartheta_1(v-s)\,\vartheta_3(e)}$ $\left(e \neq \frac{1+\tau}{2} \bmod \mathbb{Z}+\tau\mathbb{Z}\right)$

$$\det\left[K(v_i,s_j)\right]_{i,j=1}^{N} = \frac{\prod_{i<j} \vartheta_1(v_i-v_j)\,\vartheta_1(s_i-s_j)}{\prod_{i,j} \vartheta_1(v_i-s_j)} \frac{\vartheta_3\left(\sum(v_i-s_i)+e\right)}{\vartheta_3(e)}$$

How to prove it?:

-) Both sides are antisymmetric in exchanges $s_i \leftrightarrow s_j$ or $v_i \leftrightarrow v_j \Rightarrow$ study as function of v_1
-) Both sides have zeros when $v_1 \in \{v_2, \dots v_N\}$, poles when $v_1 \in \{s_1, \dots s_N\}$
-) Both sides have the same quasi-periodicity under shifts $v_1 \mapsto v_1 + 1$, $v_1 \mapsto v_1 + \tau$
-) $\dfrac{\text{LHS}}{\text{RHS}}$ must be elliptic and can have at most 1-pole $\Rightarrow$ (R.R.) <u>constant</u>
-) By coalescence, figure out constant $=1$.

More about Δ's (a.k.a. 2-Torsion points of $\mathbb{J}(C)$)

The half periods of $\mathbb{J}(C)$ are in 1-1 correspondence with semi-canonical line bundles namely, line bundles whose square is K (holomorphic differentials) $\to$ Spin bundles.

•) the even have no holomorphic sections, Generically (in the moduli space $\mathcal{M}_g$)

••) the nonsingular odd have exactly one hol. section

what is that?

Δ odd-nonsingular $\longleftrightarrow$ $\mathcal{D}_\Delta$ of degree $g-1$
(Riemann, see above)

Then $2\mathcal{D}_\Delta = K$, namely there is a holomorphic differential ω_Δ such that $\operatorname{div} \omega_\Delta = 2\mathcal{D}_\Delta$ and then

$$h_\Delta = \sqrt{\omega_\Delta}$$ is a well-defined spinor (half-form)

$$\boxed{\text{Formula} \quad \omega_\Delta(p) = \sum_{l=1}^{g} \left(\frac{\partial}{\partial z_l} \Theta_\Delta \Big|_{z=0} \right) \cdot \omega_j(p)}$$

(Fay '73)

Fundamental bidifferential (a.k.a. "Bergman")

Take Θ_Δ as before (non-singular odd characteristics)

Define

differential w.r.t. variable q or p, respectively

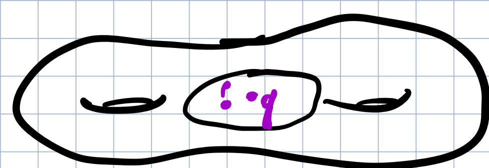
$$\Omega(p,q) = d_p \, d_q \ln \Theta_\Delta\left(\mathcal{A}(p) - \mathcal{A}(q)\right)$$

Properties (exercise) (see extended notes)

① It is a bi-differential (i.e. a differential w.r.t. both variables: think $\dfrac{dz\, dw}{(z-w)^2}$)

② Symmetry: $\Omega(p,q) = \Omega(q,p)$

③ It has a double pole (w.r.t. p) for $p=q$ and nowhere else. (Use $\Theta_\Delta(\mathcal{A}(p)-\mathcal{A}(q)) = 0$ for $p=q$)



④ Normalization

$$\oint_{p\in a_j} \Omega(p,q) \equiv 0 \quad ; \quad \oint_{p \in \beta_j} \Omega(p,q) = 2\pi i \, \omega_j(q)$$

(Use periodicity properties of Θ_Δ)

⑤ In local coordinate $z = \zeta(p)$, $w = \zeta(q)$ in a same neighbourhood

$$\Omega(p,q) = \left[\frac{1}{(z-w)^2} + \frac{1}{6} S_B(w) + O(z-w) \right] dz\, dw$$

Bergman projective (affine) connection (i.e. stress-energy tensor in c=1 CFT)

$$S_B(\tilde{w}) \left(\frac{d\tilde{w}}{dw}\right)^2 = S_B(w) + \{\tilde{w}, w\}$$

Schwarzian derivative

$$\{\tilde{w}, w\} = \left(\frac{\tilde{w}''}{\tilde{w}'}\right)' - \frac{1}{2} \left(\frac{\tilde{w}''}{\tilde{w}'}\right)^2 \qquad \left(\tilde{w}' = \frac{d\tilde{w}}{dw}\right)$$

Fuchsian representation & $\dim(\mathcal{M}_g)$

There is a different representation of RSs as quotient of their universal cover by the action of a discrete group. This is entirely akin to the case of elliptic curves $\mathcal{E}_\tau \simeq \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ (the group is $\mathbb{Z}\times\mathbb{Z} \simeq \pi_1(\mathcal{E})$)

Facts

•) Any $\mathcal{C}$ of $g \geq 2$ admits a unique metric in the same conformal class of constant gaussian curvature $=-1$.

$$ds^2 = \rho(z,\bar{z})\,|dz|^2 \qquad \left(|dz|^2 = dx^2+dy^2\right)$$

$$\rho(z,\bar{z}) > 0$$

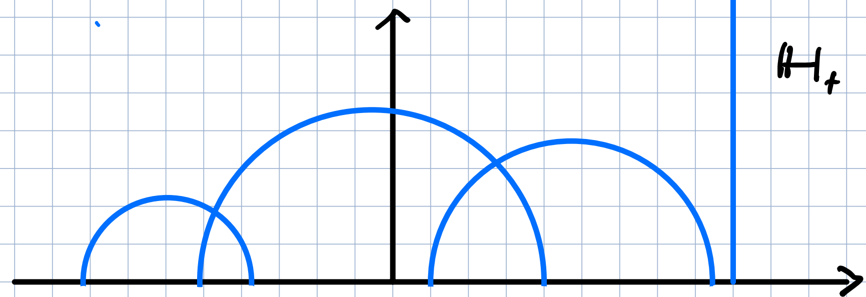
$$\Delta \ln \rho = -1$$

•) Thus the universal cover is a simply connected surface (open) with a negative curvature metric. I.e $\mathbb{H}_+ = \{\Im m\, z > 0\}$ with

$$\boxed{ds^2 = \frac{dx^2+dy^2}{y^2}}$$

•) Examples of geodesics:

semicircles with center on $\mathbb{R}$ (or vertical Lines)



•) The action of deck transformations is an isometry:

$$\mathrm{Iso}(\mathbb{H}_+, ds^2) = \mathbb{P}SL_2(\mathbb{R})$$

$$\gamma(z) = \frac{az+b}{cz+d} \qquad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \qquad a, b, c, d \in \mathbb{R}$$

Namely $\pi_1(C, p_0) \simeq \langle \alpha_1^{\pm 1}, \dots \beta_g^{\pm 1} \mid \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdot \dots \cdot \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} = id \rangle$

is represented by a (discrete) subgroup of $\mathbb{P}SL_2(\mathbb{R})$, i.e. $2g$ matrices

$\alpha_j \longmapsto A_j$; $\beta_j \longmapsto B_j \in SL_2(\mathbb{R})$ subject to

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdot \dots \cdot A_g B_g A_g^{-1} B_g^{-1} = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

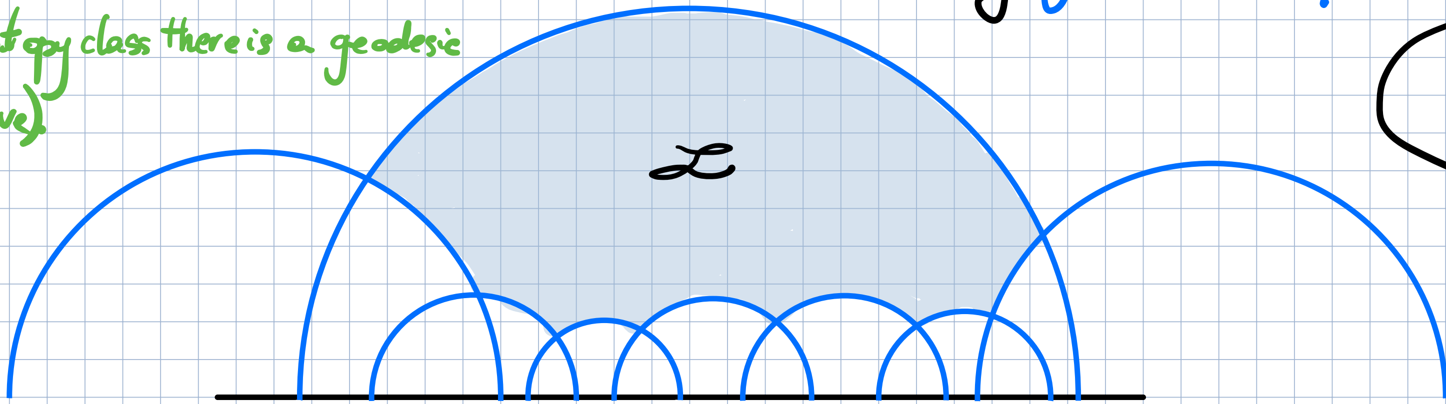
$|\mathrm{tr}\, M| > 2$

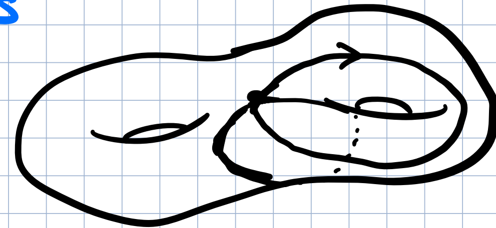
Lesson: $C \simeq \mathbb{H}_+ / \Gamma$ where Γ is a discrete group of $\overbrace{\text{(hyperbolic)}}$ isometries of $\mathbb{H}_+$

Remark $M \in PSL_2(\mathbb{R})$ is
$$\begin{cases} \text{hyperbolic} & |\mathrm{tr} M| > 2 \\ \text{elliptic} & |\mathrm{tr} M| < 2 \longrightarrow \text{nodes \& singularities} \\ \text{parabolic} & |\mathrm{tr} M| = 2 \longrightarrow \text{removed points.} \end{cases}$$

The fundamental polygon can be realized by geodesic loops

(in any homotopy class there is a geodesic representative)





However: since p_0 (basepoint) can be chosen arbitrarily (and $p_0 \mapsto \tilde{p}_0$ amounts to a conjugation) we consider the matri

Equivalently: if we conjugate all A_j, B_j's by the same $G \in \mathrm{Iso}(\mathbb{H}_+)$ we clearly have the same R.S.

$$\dim \mathcal{M}_g \;\; (= \dim \mathrm{Teich}_g)$$

What we are presenting is not just a conformal class of metrics, but also a choice of generators for π_1 (a marking).

The corresponding moduli space is called Teichmüller space.

The moduli space $\mathcal{M}_g$ is a further quotient by the action of change of basis of generators ("mapping class group"). However the dimension is the same. Let's compute it!

$$\mathcal{T}_g \simeq \mathrm{Hom}(\pi_1, \mathbb{P}SL_2^{(hyp)}(\mathbb{R})) \big/ \mathbb{P}SL_2(\mathbb{R}) = \{ \underbrace{A_1 B_1 A_1^{-1} B_1^{-1} \cdot \ldots \cdot A_g B_g A_g^{-1} B_g^{-1}}_{\text{(all hyperbolic)}} = \pm \mathbb{1} \} \big/ \mathbb{P}SL_2(\mathbb{R})$$

$$\boxed{\dim_{\mathbb{R}} \mathcal{T}_g =} \ 3 \cdot 2g - 3 - 3 = \boxed{6g-6} \quad (\text{i.e. } 3g-3 \text{ complex})$$

THE END