Theta functions

Motivation/mantra: for periodic functions, we can express all in terms of sin/cos.
For Riemann Surfaces we have Θ. "Thrigonometry"

Def Let $\tau \in Mat_{g\times g}(\mathbb{C})$, $\tau = \tau^t$ (symmetric)

$\operatorname{Im}\tau > 0$ (+ve definite)

Let $\underline{z} \in \mathbb{C}^g$ and define (Riemann Θ function)

$$\Theta(\underline{z};\tau) = \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left(i\pi \vec{n}^t \cdot \tau \cdot \vec{n} + 2\pi i\, \underline{n}^t \cdot \underline{z}\right)$$

(Fourier poly-series)

Exercise: Each term in the sum satisfies

$$|A| \leq e^{-\pi \vec{n}^t \cdot (\operatorname{Im}\tau) \cdot \vec{n} - 2\pi \underline{n}^t \cdot \operatorname{Im} \underline{z}}$$

uniformly for $\underline{z} \in$ compact sets.

Prove convergence. (Hint: M-series test or Lebesgue dominated convergence)

Remark In $g=1$ this is one of Jacobi's ϑ. (ϑ_3 in DLMF, up to normalization of arg.)

Properties

•) $\Theta(\underline{z};\Pi)$ is entire w.r.t. $\underline{z}$

••) **(Pseudo-) periodicity**. Let $\underline{\mu}, \underline{\lambda} \in \mathbb{Z}^g$

$$\Theta(\underline{z}+\underline{\mu}+\Pi\cdot\underline{\lambda}) = e^{-i\pi\underline{\lambda}^\tau\cdot\Pi\underline{\lambda} - 2\pi i\underline{\lambda}^\tau\cdot\underline{z}}\,\Theta(\underline{z})$$

(Exercise: Hint prove first for $\underline{\mu}=\underline{e}_j$ $\underline{\lambda}=0$, then $\underline{\mu}=0$ & $\underline{\lambda}=\underline{e}_j$)

Note: In particular, $\Theta(\underline{z})$ is periodic in the "real" directions, $\underline{z} \longmapsto \underline{z}+\underline{\mu}$, $\underline{\mu}\in\mathbb{Z}^g$.

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The main use is in conjunction with Abel's map:

Main Example (see Abel's theorem)

Let $\mathcal{D} = \underbrace{(p_1)+\cdots+(p_k)}_{\mathcal{D}_+} \underbrace{-(q_1)-\cdots-(q_k)}_{\mathcal{D}_-}$ be a principal divisor, i.e. (Abel)

$$\mathcal{A}(\mathcal{D}_+)-\mathcal{A}(\mathcal{D}_-) = \vec{m}+\Pi\cdot\vec{n}$$

(for some $\vec{m},\vec{n}\in\mathbb{Z}^g$)

Then the function f such that $\operatorname{div} f = \mathcal{D}$ is given as follows:

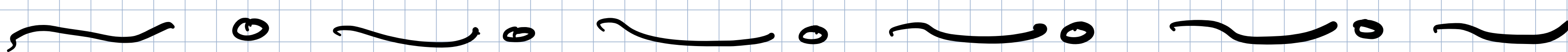
Choose $\underline{e}\in\mathbb{C}^g$: $\Theta(\underline{e})=0$ and $\nabla\Theta(\underline{e})\neq\underline{0}$

$$f(p) := \frac{\prod_{j=1}^{k}\Theta\left(\mathcal{A}(p)-\mathcal{A}(p_j)+e\right)}{\prod_{j=1}^{k}\Theta\left(\mathcal{A}(p)-\mathcal{A}(q_j)+e\right)}\cdot e^{2\pi i\,\underline{n}^{T}\cdot\mathcal{A}(p)}$$

up to multiplicative constant.

•) single-valued (Exercise: show $f(p+\alpha_j)=f(p)$
$f(p+\beta_j)=f(p)$)

••) has zeros exactly at $p=p_j$, poles exactly at $p=q_j$ and nowhere else.



The crucial theorem behind the above formula

Let $\mathfrak{f} \in \mathbb{C}^g$ generic. Then:

•) The "function" $F(p) \mapsto \Theta(\mathcal{A}(p) - \mathfrak{f})$ has g zeroes, forming a divisor $\mathcal{D}_{\mathfrak{f}}$ (of degree g)

Nota bene: $F(p+a_j) = F(p)$; $F(p+b_j) = F(p)\, e^{-i\pi \Pi_{jj} + 2\pi i (\mathcal{A}_j(p) - \mathfrak{f}_j)}$

The value of $F(p)$ is not well-defined on $\mathcal{C}$, but zeroes are.

••) The above divisor $\mathcal{D}_{\mathfrak{f}}$ (of degree g) is determined (via Jacobi Inversion Theorem) by the formula

$$\mathcal{A}(\mathcal{D}_{\mathfrak{f}}) = \mathfrak{f} + \mathbb{K}$$

(g equations for g unknowns)

where $\mathbb{K} \in \mathbb{C}^g$ is called "vector of Riemann constants" (explicit formula, but unpleasant.) and depends only on:

- the basepoint p_0 of Abel's map
- the choice of a, b's. (of course, does not depend on $\mathfrak{f}$)

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Remark: We can convey the same info as follows:

the multiplicatively multivalued function

$$F_{\mathcal{D}_g}(p) := \Theta\Big(\mathcal{A}(p) - \mathcal{A}(\underbrace{p_1 + \cdots + p_g}_{\text{zeroes}}) - \mathbb{K}\Big)$$

has $\quad \operatorname{div}\big(F_{\mathcal{D}_g}\big) = \mathcal{D}_g$

Corollary/Observation: take $p \to p_g$ in the above: then

$$0 = \Theta\big(-\mathcal{A}(p_1 + \cdots + p_{g-1}) - \mathbb{K}\big) = \Theta\big(\mathcal{A}(p_1 + \cdots + p_{g-1}) + \mathbb{K}\big)$$

For any choice of $p_1, \dots, p_{g-1}$!!

Θ is an even function: $\Theta(-z) = \Theta(z)$

Remark The Abel map & $\mathbb{K}$ depend on the basepoint p_0.

One can verify that, for divisors of degree $\boxed{g-1}$

$\mathcal{A}_{p_0}(\mathcal{D}_{g-1}) + \mathbb{K}_{p_0}$ is independent of p_0

(requires the explicit formula for $\mathbb{K}_{p_0}$)

Consequences

① Parametrization of Θ-divisor in $\mathbb{J}(\mathcal{C})$

The Θ-divisor is simply the subvariety (hypersurface) $\Theta(\underline{z}) = 0, \quad \underline{z} \in \mathbb{J}(\mathcal{C})$

Nota bene: Θ is not a single-valued function on $\mathbb{J}(\mathcal{C})$:

$$\Theta\left(\underline{z} + \underline{\mu} + \Pi \cdot \underline{\lambda}\right) = e^{-i\pi \underline{\lambda}^T \cdot \Pi \underline{\lambda} - 2\pi i \underline{\lambda}^T \cdot \underline{z}}\, \Theta(\underline{z})$$

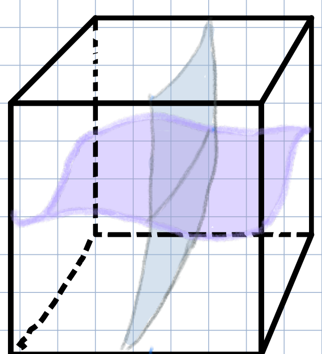
However the zero locus is well-defined because of I

Prop $\quad \Theta(\underline{e}) = 0 \quad$ iff $\quad \underline{e} = \mathcal{A}(\mathcal{D}_{g-1}) + \mathbb{K}$

In words: the Θ divisor is parametrized by $g-1$ points on $\mathcal{C}$ (possibly repeated)

Note: correct "dimension".

Remark: The Θ divisor is a central object in the algebraic geometry of Abelian varieties. It is a subvariety with singularities at locus where $i(\mathcal{D}_{g-1}) \geq 2$ ("special divisor")



Depiction of $\mathbb{J}(\mathcal{C})$ of $g=3$. Here "real" dimensions represent "complex" dimensions. It should be interpreted with periodic b.c.

gradient   Dini's Theorem!

Note: the smooth part of $\{\Theta = 0\}$ is where $\nabla\Theta \neq \underline{0}$
and it corresponds to $i(\mathcal{D}_{g-1}) = 1$ (non special, generic case)

In general (Riemann Theorem) the order of vanishing at $f = \mathcal{A}(\mathcal{D}_{g-1}) + \mathbb{K}$ is precisely $i(\mathcal{D}_{g-1}) \geq 1$. Singular locus of div Θ is in correspondence with $i(\mathcal{D}_{g-1}) \geq 2$.

Exercise Let $f \in \{\Theta(f) = 0 ; \nabla\Theta(f) \neq \underline{0}\}$ (smooth part of Θ divisor)

. Let $\mathcal{D}_{g-1}$ be the corresponding divisor, $\mathcal{A}(\mathcal{D}_{g-1}) = f + \mathbb{K}$

Show:

-) $\Theta(\mathcal{A}(p) - \mathcal{A}(q) - f)$ vanishes for $p = q$, $p \in \mathcal{D}_{g-1}$

-) $\dfrac{\Theta(\mathcal{A}(p) - \mathcal{A}(q_1) - f)}{\Theta(\mathcal{A}(p) - \mathcal{A}(q_2) - f)}$ has a simple zero at $p = q_1$, a simple pole at $p = q_2$ and no other zeroes or poles

Hint: need to show that $g-1$ zeroes in the numerator/denominator simplify.

-) Prove the formula in the Main Example, above.

Use of Θ functions.

They are used to construct canonical objects:

- Cauchy kernels (to solve boundary value problems)
- Fundamental bidifferential (a.k.a. "Bergman kernel") wrong name
  Chekhov-Eynard-Orantin topological recursion
- Projective connections (a.k.a. opers $\leftrightarrow$ BPS states)
- Szegö kernels ($\to$ det. of $\partial\bar\partial$ operators)

Θ-functions satisfy many functional identities, almost all of which reducible to **Fay-Identities.**

Θ-functions with characteristics

(Generalization of Jacobi's elliptic ϑ_j). For $\vec{n}, \vec{m} \in \mathbb{Z}^g$ one looks at the half-periods:

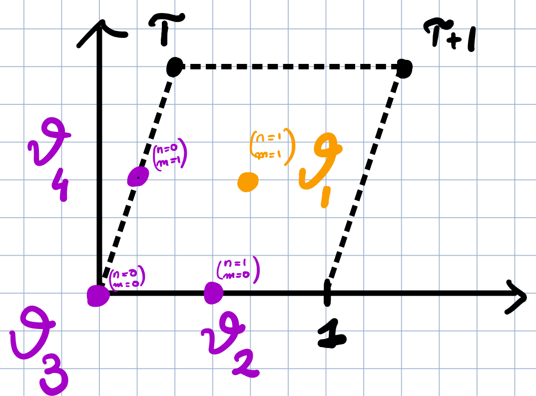
$$[n,m] = \Delta = \Delta_{\vec{n},\vec{m}} = \frac{\vec{n}}{2} + \tau \cdot \frac{\vec{m}}{2}.$$

Def $$\Theta_\Delta(\underline{z}) := \exp\left[2\pi i\left(\frac{1}{8}\vec{m}^t \tau \vec{m} + \frac{1}{2}\vec{m}\cdot\underline{z} + \frac{1}{4}\vec{m}\cdot\vec{n}\right)\right]\Theta(\underline{z}+\Delta)$$

Property $$\Theta_\Delta(-z) = e^{i\pi \vec{n}\cdot\vec{m}}\,\Theta_\Delta(z)$$

Thus we split the half periods into even/odd depending on the parity of $\vec{n}\cdot\vec{m}$.

Example (genus 1) $\mathbb{T}(\tau) = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ $(\Im m\,\tau > 0)$



If $\Delta = \frac{a}{2} + \tau\frac{b}{2}$, $a, b \in \{0,1\}$ then

$$\Theta_\Delta(z) = \sum_{n\in\mathbb{Z}+\frac{b}{2}} \exp\left(i\pi n^2\tau + 2\pi i n\left(z+\frac{a}{2}\right)\right)$$

$\Theta_{[0,0]} = \vartheta_3$ $\Theta_{[1,0]} = \vartheta_2$ $\Theta_{[0,1]} = \vartheta_4$ $\Theta_{[1,1]} = \vartheta_1$

Note: For Δ an odd half-period we necessarily have $\Theta_\Delta(\underline{0}) = 0 \Rightarrow \Delta \in \operatorname{div}(\Theta)$

There exists an (Mumford) odd, nonsingular, half period namely $\Delta = \frac{\vec{n}}{2} + \tau \cdot \frac{\vec{m}}{2}$ ($\vec{n}, \vec{m} \in \{0,1\}^g$), $\vec{n} \cdot \vec{m} \in 2\mathbb{Z}+1$ (odd) such that:

$$\text{(gradient!)} \rightarrow \nabla \Theta(\Delta) \neq \underline{0}$$

Fay identities

Let $p_1, \dots p_N, q_1, \dots q_N \in C$, Let $\underline{e} \in J(C) \setminus \operatorname{div}(\Theta)$

Then

$$\det\left[\frac{\Theta(A(p_i) - A(q_j) + \underline{e})}{\Theta(e)\, \Theta_\Delta(A(p_i) - A(q_j))}\right]_{i,j=1}^{N} = \frac{\prod_{i<j} \Theta_\Delta(A(p_i) - A(p_j))\, \Theta_\Delta(A(q_i) - A(q_j))}{\prod_{i,j=1}^{N} \Theta_\Delta(A(p_i) - A(q_j))} \cdot \frac{\Theta\left(A\left(\sum_i^N (p_i - q_i)\right) + e\right)}{\Theta(e)}$$

(For $N=3 \rightarrow$ Fay trisecant id.)

Remark

This is a higher genus generalization of Cauchy's determinantal identity

$$\det\left[\frac{1}{x_i - y_j}\right] = \frac{\prod_{i<j} (x_i - x_j)(y_i - y_j)}{\prod_{i,j} (x_i - y_j)}$$

Examples/exercises.

In genus $g=1$: There is only **one** odd characteristic $[1,1] \to$ Jacobi $\vartheta_1(z)$

Then Fay identities read: $\mathcal{K}(v,s) := \dfrac{\vartheta_3(v-s+e)}{\vartheta_1(v-s)\,\vartheta_3(e)}$ $\left(e \neq \frac{1+\tau}{2} \bmod \mathbb{Z}+\tau\mathbb{Z}\right)$

$$\det\left[\mathcal{K}(v_i,s_j)\right]_{i,j=1}^{N} = \frac{\prod_{i<j}\vartheta_1(v_i-v_j)\,\vartheta_1(s_i-s_j)}{\prod_{i,j}\vartheta_1(v_i-s_j)}\;\frac{\vartheta_3\left(\sum(v_i-s_i)+e\right)}{\vartheta_3(e)}$$

How to prove it?:

- Both sides are antisymmetric in exchanges $s_i \leftrightarrow s_j$ or $v_i \leftrightarrow v_j$ $\Rightarrow$ study as function of v_1
- Both sides have zeros when $v_1 \in \{v_2, \dots v_N\}$, poles when $v_1 \in \{s_1, \dots s_N\}$
- Both sides have the same quasi-periodicity under shifts $v_1 \mapsto v_1 + 1$, $v_1 \mapsto v_1 + \tau$
- $\dfrac{\text{LHS}}{\text{RHS}}$ must be elliptic and can have at most 1-pole $\Rightarrow$ (R.R.) <u>constant</u>
- By coalescence, figure out constant $= 1$.

More about Δ's (a.k.a. 2-Torsion points of $\mathbb{J}(C)$)

The half periods of $\mathbb{J}(C)$ are in 1-1 correspondence with semi-canonical line bundles namely, line bundles whose square is K (holomorphic differentials) $\to$ spin bundles.

•) the even have no holomorphic sections, Generically (in the moduli space $\mathcal{M}_g$)

••) the nonsingular odd have exactly one hol. section

what is that?

Δ odd-nonsingular $\longleftrightarrow$ $\mathcal{D}_\Delta$ of degree $g-1$ (Riemann, see above)

Then $2\mathcal{D}_\Delta = K$, namely there is a holomorphic differential ω_Δ such that $\mathrm{div}\,\omega_\Delta = 2\mathcal{D}_\Delta$ and then

$$h_\Delta = \sqrt{\omega_\Delta}$$ is a well-defined spinor (half-form) "primitive Δ-spinor"

$$\boxed{\textbf{Formula} \qquad \omega_\Delta(p) = \sum_{l=1}^{g} \left(\frac{\partial}{\partial z_l} \Theta_\Delta \Big|_{z=0} \right) \cdot \omega_j(p)}$$

(Fay 73)

Fundamental bidifferential (a.k.a. "Bergman")

Take Θ_Δ as before (non-singular odd characteristics)

<u>Define</u>

differential w.r.t. variable q or p, respectively

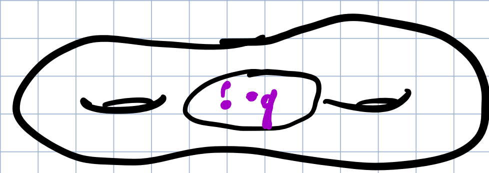
$$\Omega(p,q) = d_p \, d_q \ln \Theta_\Delta \big(A(p) - A(q)\big)$$

Properties (exercise) (see extended notes)

1. It is a bi-differential (i.e. a differential w.r.t. both variables: think $\dfrac{dz\,dw}{(z-w)^2}$)

2. Symmetry: $\Omega(p,q) = \Omega(q,p)$

3. It has a double pole (w.r.t. p) for $p=q$ and nowhere else. (Use $\Theta_\Delta(A(p)-A(q))=0$ for $p=q$)



4. Normalization

$$\oint_{p\in a_j} \Omega(p,q) \equiv 0 \quad ; \quad \oint_{p\in \beta_j} \Omega(p,q) = 2\pi i \, \omega_j(q)$$

(Use periodicity properties of Θ_Δ)

Primitive Δ-spinor & (Klein) prime form.

With the established notation the Klein prime form is

$$E(p,q) = \frac{\Theta_\Delta\left(\mathcal{A}(p)-\mathcal{A}(q)\right)}{h_\Delta(p)\ \ h_\Delta(q)} = \frac{\Theta_\Delta\left(\int_q^p \vec{\omega}\right)}{h_\Delta(p)\, h_\Delta(q)}$$

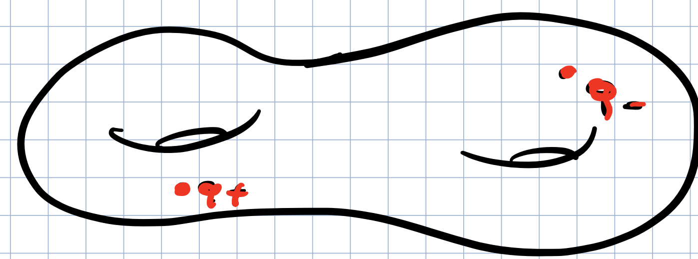
Properties:

1. $E(p,q) = -E(q,p)$
2. $E(p+a_j,q) = E(p,q)$
3. $E(p+b_j,q) = e^{-i\pi \Pi_{jj} - 2i\pi \int_p^q \omega_j} E(p,q)$
4. Vanishes for $p=q$ and nowhere else (¿ how comes?)
5. In local coordinate ζ, setting $z=\zeta(p)$ $w=\zeta(q)$, we have

$$E(p,q) = \frac{\mathbb{E}(z,w)}{\sqrt{dz}\ \sqrt{dw}} = \frac{(z-w)}{\sqrt{dz}\ \sqrt{dw}}\left(1+\mathcal{O}((z-w)^2)\right) \quad \text{(normalization)}$$

Does not depend on choice of Δ!

3^{rd} kind differentials



Fix $q_+, q_- \in \mathcal{C}$

Consider the object

$$\Omega_{q_+ q_-}(p) = d_p \ln \frac{\Theta_\Delta(A(p) - A(q_+))}{\Theta_\Delta(A(p) - A(q))}$$

Exercise:

- $\Omega_{q_+ q_-}$ has only two simple poles at q_+, q_-
- $\operatorname{res}_{p = q_\pm} \Omega_{q_+ q_-}(p) = \pm 1$
- single-valued
- $\oint_{a_j} \Omega_{q_+ q_-} = 0 \quad \forall j = 1 .. g$
- $\int_{p_-}^{p_+} \omega_{q_+ q_-} = \int_{q_-}^{q_+} \omega_{p_+ p_-}$ (exchange formula $\longleftrightarrow$ "cross ratio")

$\longleftrightarrow$ Weyl reciprocity.

Fun with Fay

One can use the prime form $E(p,q)$ to construct "Szegö kernels". Fix $\underline{e} \notin \mathrm{div}(\Theta)$ $\mathbb{J}_g$

$$S(p,q) := \frac{\Theta(\mathcal{A}(p)-\mathcal{A}(q)-\underline{e})}{\Theta(\underline{e})\; E(p,q)} = \frac{\Theta(\mathcal{A}(p)-\mathcal{A}(q)-\underline{e})}{\Theta(\underline{e})\;\Theta_\Delta(\mathcal{A}(p)-\mathcal{A}(q))}\, h_\Delta(p)\, h_\Delta(q)$$

- bi-spinor
- Simple pole only on diagonal, "residue" 1

$$\left(\frac{1}{(z-w)}+\dots\right)\sqrt{dz}\sqrt{dw}$$

- section of flat bundle $\chi \otimes \chi^{-1}$ (minor modifications, we can make it a $U(1)$ bundle)

Fay take 2

$$\det\left(S(p_i,q_j)\right)_{i,j=1}^{N} = \frac{\prod_{i<j}\Theta_\Delta(\mathcal{A}(p_i)-\mathcal{A}(p_j))\,\Theta_\Delta(\mathcal{A}(q_i)-\mathcal{A}(q_j))}{\prod_{i,j=1}^{N}\Theta_\Delta(\mathcal{A}(p_i)-\mathcal{A}(q_j))}\;\frac{\Theta\left(\mathcal{A}\left(\sum_{i}^{N}p_i-q_i\right)-\underline{e}\right)}{\Theta(\underline{e})}$$

Proposition (Dubrovin '12, can be proved from degenerating Fay see Eynard-Borot)

$$\sum_{\mathbf{j}\in[1,\dots,g]^N} \frac{\partial^N \ln \Theta(-\underline{e})}{\partial_{j_1}\cdots\partial_{j_N}}\,\omega_{j_1}(p_1)\cdots\omega_{j_N}(p_N) =$$

$$= \frac{(-1)^{N-1}}{N}\sum_{\sigma\in S_N}\overbrace{S(p_{\sigma_1},p_{\sigma_2})S(p_{\sigma_2},p_{\sigma_3})\cdots S(p_{\sigma_{N-1}},p_{\sigma_N})S(p_{\sigma_N},p_{\sigma_1})}^{N\text{-terms}} + \delta_{N,2}B(p_1,p_2)$$

perm group

Useful in all manners of computations of correlators of KP Tau-functions...

Remark: For $N=2$ the formula is in Fay '73

Fuchsian representation & $\dim(\mathcal{M}_g)$

There is a different representation of RSs as quotient of their universal cover by the action of a discrete group. This is entirely akin to the case of elliptic curves $\mathcal{E}_\tau \simeq \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ (the group is $\mathbb{Z}\times\mathbb{Z} \simeq \pi_1(\mathcal{E})$)

Facts

•) Any $\mathcal{C}$ of $g \geq 2$ admits a unique metric in the same conformal class of constant gaussian curvature $=-1$.

$$ds^2 = \rho(z,\bar{z})\,|dz|^2 \qquad \left(|dz|^2 = dx^2+dy^2\right)$$

$$\rho(z,\bar{z}) > 0$$

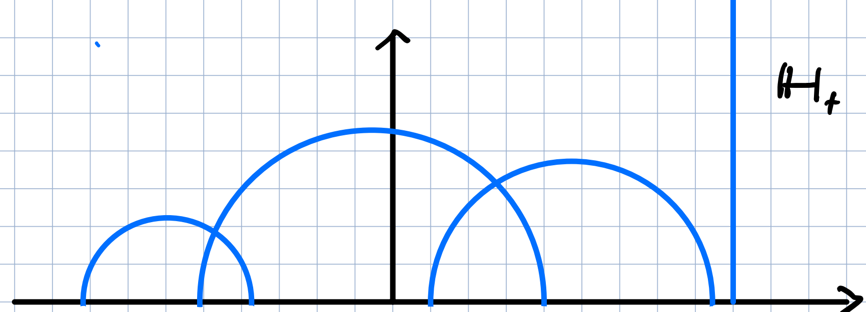
$$\Delta \ln \rho = -1$$

•) Thus the universal cover is a simply connected surface (open) with a negative curvature metric. I.e. $\mathbb{H}_+ = \{\Im m\, z > 0\}$ with

$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

•) Examples of geodesics:

semicircles with center on $\mathbb{R}$ (or vertical lines)



•) The action of deck transformations is an isometry:

$$\mathrm{Iso}(\mathbb{H}_+, ds^2) = \mathbb{P}SL_2(\mathbb{R})$$

$$\gamma(z) = \frac{az+b}{cz+d} \qquad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \qquad a, b, c, d \in \mathbb{R}$$

Namely $\pi_1(C, p_0) \simeq \langle \alpha_1^{\pm 1}, \dots \beta_g^{\pm 1} \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdot \dots \cdot \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = id \rangle$

is represented by a (discrete) subgroup of $\mathbb{P}SL_2(\mathbb{R})$, i.e. $2g$ matrices

$\alpha_j \longmapsto A_j \ ; \ \beta_j \longmapsto B_j \in SL_2(\mathbb{R})$ subject to

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdot \dots \cdot A_g B_g A_g^{-1} B_g^{-1} = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad |\mathrm{tr}\, M| > 2$$

Lesson: $C \simeq \mathbb{H}_+ / \Gamma$ where Γ is a discrete group of (hyperbolic) isometries of $\mathbb{H}_+$

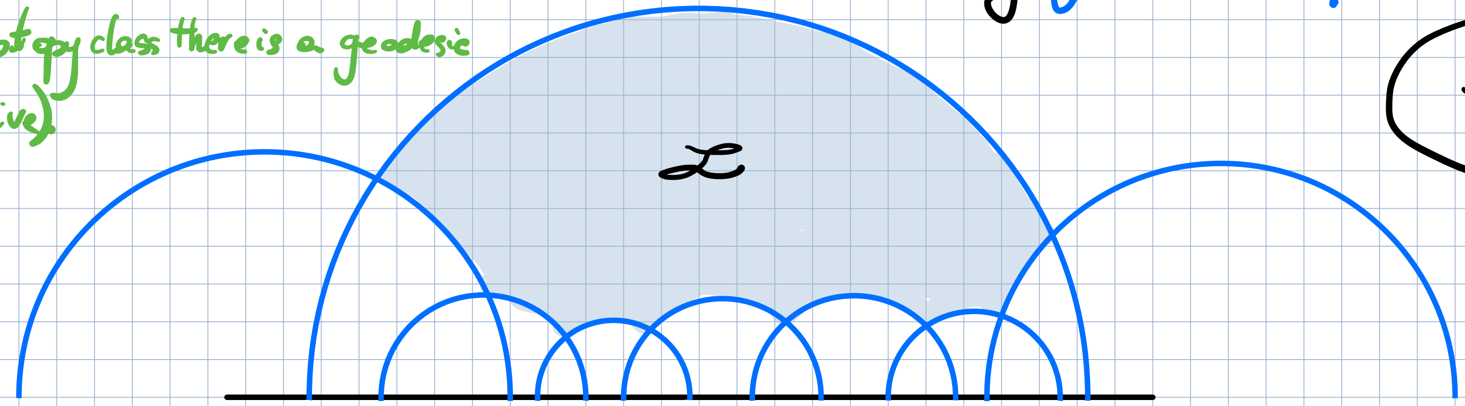
Remark $M \in PSL_2(\mathbb{R})$ is $\begin{cases} \text{hyperbolic} & |\mathrm{tr} M| > 2 \\ \text{elliptic} & |\mathrm{tr} M| < 2 \\ \text{parabolic} & |\mathrm{tr} M| = 2 \end{cases}$

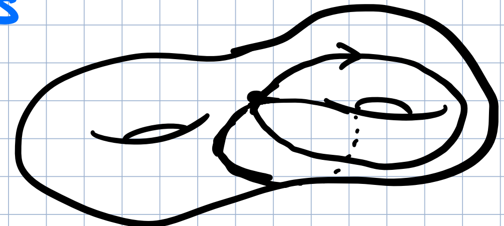
elliptic $\longrightarrow$ nodes & singularities

parabolic $\longrightarrow$ removed points.

The fundamental polygon can be realized by geodesic loops

(in any homotopy class there is a geodesic representative)





However: since p_0 (basepoint) can be chosen arbitrarily (and $p_0 \mapsto \tilde{p}_0$ amounts to a conjugation) we consider the matri

Equivalently: if we conjugate all A_i, B_i's by the same $G \in \mathrm{Iso}(\mathbb{H}_+)$ we clearly have the same R.S.

$$\dim \mathcal{M}_g \; (= \dim \mathrm{Teich}_g)$$

What we are presenting is not just a conformal class of metrics, but also a choice of generators for π_1 (a marking).

The corresponding moduli space is called Teichmüller space.

The moduli space $\mathcal{M}_g$ is a further quotient by the action of change of basis of generators ("mapping class group"). However the dimension is the same. Let's compute it!

$$\mathcal{T}_g \simeq \mathrm{Hom}(\pi_1, PSL_2^{(hyp)}(\mathbb{R}))/PSL_2(\mathbb{R}) = \{A_1 B_1 A_1^{-1} B_1^{-1} \cdot \ldots \cdot A_g B_g A_g^{-1} B_g^{-1} = \pm 1\}/PSL_2(\mathbb{R})$$

(all hyperbolic)

$$\boxed{\dim_{\mathbb{R}} \mathcal{T}_g =} \; 3 \cdot 2g - 3 - 3 = \boxed{6g-6}$$

(i.e. $3g-3$ complex)

THE END