MODULI SPACES OF RIEMANN SURFACES – EXERCISES

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Lecture 1: moduli spaces of Riemann surfaces and their stratification

Exercise 1.

(1) *Consider a genus* 0 *curve with three marked points* $(\mathbb{P}^1, p_1, p_2, p_3)$. *Find the (unique)* $g \in \mathrm{PSL}(2,\mathbb{C})$ *that maps* $(\mathbb{P}^1, p_1, p_2, p_3)$ *to* $(\mathbb{P}^1, 0, 1, \infty)$.

(2) *Consider a genus* 0 *curve with four marked points* $(\mathbb{P}^1, p_1, p_2, p_3, p_4)$. *The element* $g \in \mathrm{PSL}(2,\mathbb{C})$ *found in part (1) maps* $(\mathbb{P}^1, p_1, p_2, p_3, p_4)$ *to* $(\mathbb{P}^1, 0, 1, \infty, t)$. *Find an expression for* t *as a function of* p_1, p_2, p_3, p_4.

Exercise 2. *For the reader familiar with Riemann–Roch and Riemann–Hurwitz, convince yourself that the complex dimension of* $\mathcal{M}_g = \mathcal{M}_{g,0}$ *is* $3g-3$. *To this end, consider the moduli space of pairs* (Σ, f), *where* Σ *is a genus* g *Riemann surface and* f *is a degree* d *holomorphic map from* Σ *to* $\mathbb{P}^1$ *(i.e. a meromorphic function on* X*). Such a space is sometimes referred to as a Hurwitz space, denoted* $\mathcal{H}_{g,d}$. *Compute its dimension in two different ways.*

- *The dimension of* $\mathcal{H}_{g,d}$ *equals the dimension of* $\mathcal{M}_g$, *counting the "number of deformation parameters" of the Riemann surface* Σ, *plus the "number of deformation parameters" of the function* f. *Compute the latter via Riemann–Roch.*
- *Directly compute the dimension of* $\mathcal{H}_{g,d}$ *using Riemann–Hurwitz.*

Conclude that $\dim \mathcal{M}_g = 3g-3$.

Exercise 3. *The Euler characteristic of an orbifold* X *is defined as*

$$\chi(X) = \sum_G \frac{\chi(X_G)}{|G|}\,, \tag{0.1}$$

where X_G *is the locus of points with automorphism group* G. *Prove that* $\chi(\mathcal{M}_{1,1}) = -\frac{1}{12}$.

Exercise 4.

(1) *List all strata of* $\overline{\mathcal{M}}_{2,1}$.

(2) *Consider a stable graph* Γ *of type* (g,n). *Show that the dimension of the stratum is* $\dim(\mathcal{M}_\Gamma) = \dim(\overline{\mathcal{M}}_{g,n}) - |E_\Gamma|$.

Lecture 2: Witten's conjecture

Exercise 5. *Employ the geometric string and dilaton equations, together with the projection formula and the expression $[\Gamma] = \frac{1}{|\mathrm{Aut}(\Gamma)|}\xi_{\Gamma,*}\mathbf{1}$ for the Poincaré dual of boundary strata, to prove the following equations satisfied by Witten's correlators.*

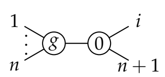
- ***String equation.*** *Integrals over $\overline{\mathcal{M}}_{g,n+1}$ with no ψ_{n+1} are reduced to integrals over $\overline{\mathcal{M}}_{g,n}$:*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \left(\prod_{j \neq i} \psi_i^{d_i} \right) \psi_i^{d_i - 1} \,. \tag{0.2}$$

In Witten's notation, the string equation amounts to the removal of a τ_0:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \tau_0 \rangle_g = \sum_{i=1}^{n} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_n} \rangle_g \,. \tag{0.3}$$

Hints.

- *By looking at cohomological degrees, what can you say about $\int_{\overline{\mathcal{M}}_{g,n+1}} \pi^* \alpha$ for $\alpha \in H^{2(3g-3+n)}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$?*
- *Let $D_i = \Big[$*  *$\Big]$. Interpreting it as a Poincaré dual, one can see that $D_i \cdot D_j = 0$ for all $i \neq j$.*

- ***Dilaton equation.*** *Integrals over $\overline{\mathcal{M}}_{g,n+1}$ with a single power of ψ_{n+1} are reduced to integrals over $\overline{\mathcal{M}}_{g,n}$:*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} \psi_{n+1} = (2g - 2 + n) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \,. \tag{0.4}$$

In Witten's notation, the string equation amounts to the removal of a τ_1:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \tau_1 \rangle_g = (2g - 2 + n) \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \,. \tag{0.5}$$

Exercise 6. *Knowing the string equation and the integral $\int_{\overline{\mathcal{M}}_{0,3}} \mathbf{1} = \langle \tau_0^3 \rangle_0 = 1$, show that all genus 0, ψ-class intersection numbers are determined. Can you prove the following closed formula:*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \binom{n-3}{d_1, \ldots, d_n} , \tag{0.6}$$

where $\binom{D}{d_1,\ldots,d_n} = \frac{D!}{d_1! \cdots d_n!}$ is the multinomial coefficient?

Exercise 7. *Knowing the string equation, the dilaton equation, and the integral $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \langle \tau_1 \rangle_1 = \frac{1}{24}$, show that all genus 1, ψ-class intersection numbers are determined. Can you prove the following closed formula:*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_1 = \frac{1}{24} \left(\binom{n}{d_1, \ldots, d_n} - \sum_{\epsilon_1, \ldots, \epsilon_n \in \{0,1\}} \binom{n - |\epsilon|}{d_1 - \epsilon_1, \ldots, d_n - \epsilon_n} (|\epsilon| - 2)! \right) , \tag{0.7}$$

where $|\epsilon| = \epsilon_1 + \cdots + \epsilon_n$?

Exercise 8. *Prove that $\langle \tau_1 \rangle_1 = \frac{1}{24}$ using the following facts.*

(1) The following identity holds for arbitrary line bundle $\mathcal{L}$: $c_1(\mathcal{L}) = \frac{1}{k} c_1(\mathcal{L}^{\otimes k})$.

(2) For an arbitrary line bundle $\mathcal{L}$, we have $c_1(\mathcal{L}) = [Z - P]$, where Z and P are the divisors of zeros and poles of a generic meromorphic section of $\mathcal{L}$ and $[\,\cdot\,]$ denotes the Poincaré dual[1].

(3) Consider the cotangent line bundle $\mathcal{L}_1^{\otimes k} \to \overline{\mathcal{M}}_{1,1}$. There is a canonical identification of the vector space of holomorphic sections of $\mathcal{L}_1^{\otimes k}$ and the vector space of modular forms of weight k.

(4) The following (combination of) Eisenstein series

$$\begin{aligned} G_4(\tau) &= \sum_{\lambda \in (\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}} \frac{1}{\lambda^4}\,, \\ G_6(\tau) &= \sum_{\lambda \in (\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}} \frac{1}{\lambda^6}\,, \\ \tilde{G}_{12}(\tau) &= \left(\frac{G_4(\tau)}{2\zeta(4)}\right)^3 - \left(\frac{G_6(\tau)}{2\zeta(6)}\right)^2, \end{aligned} \tag{0.8}$$

are modular forms of weight 4, 6, and 12 respectively. Besides, they have a unique simple zero at $\tau = \frac{1+\mathrm{i}\sqrt{3}}{2}$, $\tau = \mathrm{i}$, and $\tau = +\mathrm{i}\infty$ respectively.

Exercise 9. *Define the differential operators*

$$L_{-1} = \hbar \frac{\partial}{\partial t_0} - \hbar^2 \left(\sum_{k \geq 1} t_k \frac{\partial}{\partial t_{k-1}} + \frac{t_0^2}{2} \right), \tag{0.9}$$

$$L_0 = \hbar \frac{\partial}{\partial t_1} - \hbar^2 \left(\sum_{k \geq 0} \frac{2k+1}{3}\, t_k \frac{\partial}{\partial t_k} + \frac{1}{24} \right). \tag{0.10}$$

Prove the following:

- *The string equation and $\langle \tau_0^3 \rangle_0$ are equivalent to the equation $L_{-1}\, Z = 0$.*
- *The dilaton equation and $\langle \tau_1 \rangle_1 = \frac{1}{24}$ are equivalent to the equation $L_0\, Z = 0$.*

Exercise 10. *Prove that the collection $(L_n)_{n \geq -1}$ of differential operators defined by equation (0.9), (0.10), and*

$$L_n = \hbar \frac{\partial}{\partial t_{n+1}} - \hbar^2 \left(\sum_{k \geq 0} \frac{(2n+2k+1)!!}{(2n+3)!!(2k-1)!!}\, t_k \frac{\partial}{\partial t_{k+n}} + \frac{1}{2} \sum_{\substack{a,b \geq 0 \\ a+b=n-1}} \frac{(2a+1)!!(2b+1)!!}{(2n+3)!!} \frac{\partial^2}{\partial t_a \partial t_b} \right) \tag{0.11}$$

for $n \geq 1$ is indeed a representation of the Virasoro algebra: $[L_m, L_n] = \hbar^2 (m-n) L_{m+n}$. This, together with the form (0.11) of the operators, proves that $(L_n)_{n \geq -1}$ form an Airy ideal (see Vincent's lectures).

[1] Poincaré duality for orbifolds involves the automorphism group. More precisely, if Z is a sub-orbifold of X with underlying topological space $\hat{Z}$, then $[Z] = \frac{1}{|G|}[\hat{Z}]$, where G is the automorphism group of a generic point in $\hat{Z}$.

Exercise 11 (💀). *Show that the Virasoro constraints are equivalent to the following* topological recursion *for Witten's correlators:*

$$\begin{aligned}\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \sum_{m=2}^{n} \frac{(2d_1 + 2d_m - 1)!!}{(2d_1+1)!!\,(2d_m-1)!!} \langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle_g \\ + \frac{1}{2} \sum_{a+b=d_1-2} \frac{(2a+1)!!\,(2b+1)!!}{(2d_1+1)!!} \Bigg(\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle_{g-1} \\ + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{ d_2,\dots,d_n \}}} \langle \tau_a \tau_{I_1} \rangle_{g_1} \langle \tau_b \tau_{I_2} \rangle_{g_2} \Bigg) .\end{aligned} \tag{0.12}$$

Prove that the above recursion is equivalent to the Eynard–Orantin topological recursion formula (see Vincent's lectures) on the Airy spectral curve $(\mathbb{P}^1,\ x(z) = \frac{z^2}{2},\ y(z) = z,\ B(z_1,z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1-z_2)^2})$*:*

$$\omega_{g,n}(z_1,\dots,z_n) = (-1)^n \sum_{\substack{d_1,\dots,d_n \geq 0 \\ d_1+\cdots+d_n = 3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} \frac{(2d_i+1)!!}{z_i^{2d_i+2}} \mathrm{d}z_i \,. \tag{0.13}$$

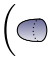
Lecture 3: cohomological field theories and topological recursion

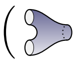
Exercise 12. *Let (V, η, e, Ω) be a CohFT with unit. Prove that $(V, \eta, e, \star)$ forms a Frobenius algebra, that is, it satisfies*

$$\eta(v_1 \star v_2, v_3) = \eta(v_1, v_2 \star v_3)\,. \tag{0.14}$$

A Frobenius algebra is equivalent to a 2D topological field theory $\mathcal{Z}$ via the following assignments: $\mathcal{Z}(S^1) = V$ for the Hilbert space of states on the circle and

$\mathcal{Z}\Big($  $\Big) = \eta\colon V \otimes V \to \mathbb{Q}\,,$

$\mathcal{Z}\Big($  $\Big) = e\colon \mathbb{Q} \to V\,,$ (0.15)

$\mathcal{Z}\Big($  $\Big) = \star\colon V \otimes V \to V\,,$

for the morphisms. The partition function $\mathcal{Z}(\Sigma_{g,n,m})$ of any genus g surfaces connecting n initial states to m final states can be reconstructed from the above values using the TFT properties.

Exercise 13. *Prove that $\exp(2\pi^2\kappa_1)$ is the CohFT obtained from the trivial one under the action of the following translation:*

$$T(u) = \sum_{k\geq 1} \frac{(-2\pi^2)^k}{k!} u^{k+1} = u\big(1 - e^{-2\pi^2 u}\big)\,. \tag{0.16}$$

Exercise 14. *Prove, using Mumford's formula, that $\Lambda(t)\Lambda(-t) = 1$. This is sometimes referred to as Mumford's relation. Deduce the relations $\lambda_g^2 = 0$.*

Exercise 15 (💀). *Show that the CohFT associated to the following spectral curve*

$$\left(\mathbb{P}^1,\ x(z) = -f\log(z) - \log(1-z),\ y(z) = -\log(z),\ B(z_1, z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1 - z_2)^2}\right)\,. \tag{0.17}$$

is the triple Hodge $\Lambda(1)\Lambda(f)\Lambda(-f-1)$. This is the CohFT underlying the (framed) topological vertex, and the topological recursion formula for the triple Hodge class is nothing but the BKMP remodelling conjecture for the vertex. The large framing limit recovers the so-called Lambert curve computing Hurwitz numbers.

💡 Hint. *Recall the integral representation of the Euler Beta function*

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \tag{0.18}$$

and the asymptotic expansion of the Euler Gamma function

$$e^{\frac{1}{v}}\sqrt{2\pi}\,\frac{(-v)^{\frac{1}{v}+a+\frac{1}{2}}}{\Gamma(a - v^{-1})} \sim \exp\left(\sum_{m=1}^{\infty} \frac{B_{m+1}(a)}{m(m+1)} v^m\right). \tag{0.19}$$

Here $B_{m+1}(a)$ are Bernoulli polynomials, and specialise to Bernoulli numbers at both $a = 0$ and $a = 1$: $B_{m+1}(0) = B_{m+1}(1) = B_{m+1}$.

Exercise 16. *Consider the spectral curve*

$$\left(\mathbb{P}^1,\ x(z) = \frac{z^2}{2},\ y(z) = \frac{\sin(2\pi z)}{2\pi z},\ B(z_1, z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1 - z_2)^2}\right)\,. \tag{0.20}$$

Using the CohFT/topological recursion correspondence and the expression for the Weil–Petersson form $\exp(2\pi^2\kappa_1)$ *in terms of Givental's action (exercise 13), show that the topological recursion correlators associated to the above spectral curve compute the differential of the Laplace transform of the Weil–Petersson volumes:*

$$\omega_{g,n}(z_1,\dots,z_n) = \mathrm{d}_{z_1}\cdots\mathrm{d}_{z_n}\left(\prod_{i=1}^{n}\int_0^\infty \mathrm{d}L_i\, e^{-z_iL_i}\right) V_{g,n}^{\mathrm{WP}}(L_1,\dots,L_n)\,. \tag{0.21}$$