Les Houches lectures on non-perturbative Seiberg-Witten geometry

Lotte Hollands

July 2024

1 Introduction

In lectures by Andrew Neitzke last week we have learned about the spectrum of BPS states in four-dimensional $\mathcal{N} = 2$ theories, and in particular those of class S. Nikita Nekrasov, on the other hand, has taught us about $\mathcal{N} = 2$ partition functions that in some sense count instanton configurations on a four-dimensional space-time. In particular, he has introduced the Ω-background, and argued that the $\mathcal{N} = 2$ instanton partition function Z_{4d}^{inst} can be computed exactly in this background for an overlapping, but complementary, class of four-diimensional $\mathcal{N} = 2$ quiver theories.

In these lectures the goal is to combine these two elements, with the aim of constructing a **new partition function** that encodes both the instanton partition function as well as the spectrum of BPS states. Spectral networks play a central role in this story, and these lectures owe much to the beautiful works of Gaiotto-Moore-Neitzke. The approach we take is also closely related to various other perspectives and results in the literature, such as the topics of non-perturbative topological string theory, resurgence, isomonodromic tau functions, analytic Langlands, Riemann-Hilbert problems, holomorphic Floer theory, etc. In particular, there is a very close connection to exact WKB analysis, a topic which Kohei Iwaki will introduce in detail this week.

The title of "non-perturbative Seiberg-Witten geometry" might be a little confusing, as you may argue that the $\mathcal{N} = 2$ instanton partition function is already a non-perturbative object in the parameters ϵ_i of the Ω-background. Yet, I want to argue that it is natural to define a new $\mathcal{N} = 2$ partition function $Z^{\vartheta}_{\mathrm{4d}}$, and corresponding Seiberg-Witten geometry, that depends in a locally constant way on an additional phase ϑ, in such a way that it naturally reproduces the instanton partition function $Z^{\mathrm{inst}}_{\mathrm{4d}}$ at the phase corresponding to the W-bosons in the theory, as well as non-pertubative versions of the topological string partition function in an opposite phase.

To make the relation to the exact WKB analysis as clear as possible, I have decided to start these lectures in two dimensions. We will thus start with analysing two-dimensional $\mathcal{N} = (2, 2)$ theories, with emphasis on the class of Landau-Ginzburg models. The latter models are closely related to minimal models, integrable hierarchies of KdV type, and matrix models, so hopefully this will also make a connection to the lectures in earlier weeks in this school.

In the first lecture I will introduce families of (so-called massive) two-dimensional $\mathcal{N} = (2, 2)$ theories and discuss their BPS soliton spectrum. Just like in four dimensions, we will see that this spectrum is invariant under small deformations, but may jump across certain walls in the parameter space C. In fact, this wall-crossing is described by the celebrated Cecotti-Vafa wall-crossing formula that Andy alluded to in his lectures, as giving the inspiration to understanding its four-dimensional analogue. We will describe how the 2d BPS solitons are encoded in spectral networks embedded in the parameter space C, and how to obtain the Cecotto-Vafa formula from this perspective (based on the works of Gaiotto-Moore-Neitzke).

In the second lecture I will turn on the Ω-background in two dimensions, depending on a single paramater ϵ, and introduce the corresponding $\mathcal{N} = (2, 2)$ partition function $Z^{\mathrm{vortex}}_{\mathrm{2d}}$. I will explain in which sense this partition function encodes BPS vortex configations in two dimensions. We will see that the Ω-background effectively quantizes the spectral geometry, determining a (generalized) Schrödinger operator on the parameter space C, for which the vortex partition function $Z^{\mathrm{vortex}}_{\mathrm{2d}}$ is naturally a solution. But, making the connection to exact WKB analysis, we will see that there is a more general partition function $Z_{2\mathrm{d}}^{\vartheta}$ that not only encodes the vortex partition function, but also encodes the 2d BPS soliton spectrum in a canonical way. This is the two-dimensional analogue of the four-dimensional case.

In the third lecture I want to turn to four-dimensional $\mathcal{N} = 2$ theories. I'll summarise some of the main points we have learned in the lectures by Andy and Nikita, and illustrate how the two-dimensional $\mathcal{N} = (2, 2)$ theories we have learned about so far, may be embedded in these 4d theories as surface defects. I will also review Andy's derivation of the four-dimensional wall-crossing formula and point out the similarity to the Cecotti-Vafa formula. This will bring us to an important point: a better understanding of the IR line defect vevs $\mathcal{X}_{\gamma}^{\vartheta}$. Here, I will not assume you understood the physics, and define these vevs from a purely mathematical perspective, called $\mathcal{W}$-abelianization.

In the fourth lecture I argue that the objects $\mathcal{X}_{\gamma}^{\vartheta}$ form an algebra that should be seen as the four-dimensional version of the chiral algebra in two dimensions. I will introduce the $\frac{1}{2}\Omega$-background, also known as the Nekrasov-Shatashvili background, and argue that this quantizes the spectrum of the above algebra. I call the resulting geometry the non-perturbative Seiberg-Witten geometry, and show that it encodes the instanton partition function $Z_{4\mathrm{d}}^{\mathrm{inst}}$ as well as the four-dimensional BPS particle spectrum.

If I have time, I will indicate the relation to the TS/ST (topological strings versus spectral theory) correspondence, developed by Marcos and his collaborators Grassi, Hatsuda, etc, by introducing a class of spectral problems that is naturally associated to $\mathcal{N} = 2$ theories of class S. I will show that their spectral determinants are related in a natural way to the generalised partition function $Z_{4\mathrm{d}}^{\vartheta}$ at an opposite phase to that of the W-bosons.

Part 1: 2d QFT's and exact WKB analysis

In this part we analyse the "non-perturbative" structure of 2d quantum field theories that are invariant under the extended $\mathcal{N} = (2, 2)$ supersymmetry algebra. To be self-contained, these notes start with summarizing some of the basic ingredients that go towards defining such $\mathcal{N} = (2, 2)$ theories. The main text starts with the introduction of chiral rings and the associated spectral geometry. Inevitably, I have forgotten to write down essential argumemts, or even made mistakes in what I have written down. Please let me know if you spot something, and I refer you to, for instance, the Mirror Symmetry book or the more recent papers of Gaiotto, Moore and Witten for many more details.

Supersymmetry algebra

The (Lorentzian) $\mathcal{N} = (2,2)$ algebra has four odd generators $Q_\pm$ and (the Hermitean conjugates) $\overline{Q}_\pm$, known as the supercharges, which obey the (non-zero) anti-commutation relations

$$\begin{aligned} &\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \\ &[iM, Q_\pm] = \mp Q_\pm, \qquad [iM, \overline{Q}_\pm] = \mp \overline{Q}_\pm, \end{aligned} \tag{1}$$

in terms of the Hamiltonian H, the momentum P and the angular momentum M. Whereas H, P and M are the Noether charges for time translations ∂_t, spatial translations ∂_σ, and Lorentz rotation $t\partial_\sigma - \sigma\partial_t$, respectively, the supercharges are the Noether charges of supersymmetry transformations.

As is common in supersymmetry algebras, we can introduce central charges Z and $\tilde{Z}$ (with complex conjugates Z^* and $\tilde{Z}^*$) in the following way

$$\begin{aligned} &\{Q_+, Q_-\} = Z^* \qquad &\{\overline{Q}_+, \overline{Q}_-\} = Z \\ &\{Q_+, \overline{Q}_-\} = \tilde{Z}^* \qquad &\{Q_-, \overline{Q}_+\} = \tilde{Z}. \end{aligned} \tag{2}$$

These commute with all other operators in the algebra.

The $\mathcal{N} = (2,2)$ algebra may have an internal R-symmetry $U(1)_L \times U(1)_R$, which rotates the supercharges. If we define

$$\begin{aligned} &U(1)_V = \text{diag}(U(1)_L \times U(1)_R), \\ &U(1)_A = \text{anti-diag}(U(1)_L \times U(1)_R), \end{aligned} \tag{3}$$

known as the vector and the axial R-symmetries, then their generators F_V and F_A act on the supercharges as

$$\begin{aligned} &[F_V, Q_\pm] = -Q_\pm, \qquad &[F_A, Q_\pm] = \mp Q_\pm, \\ &[F_V, \overline{Q}_\pm] = +\overline{Q}_\pm, \qquad &[F_A, \overline{Q}_\pm] = \pm \overline{Q}_\pm. \end{aligned} \tag{4}$$

This implies that Z has to be zero when $U(1)_V$ is conserved, and $\tilde{Z}$ has to be zero when $U(1)_A$ is conserved. We will want to preserve the $U(1)_A$ symmetry, so that $\tilde{Z}$ can be assumed to be zero.

Note that the $\mathcal{N} = (2, 2)$ algebra is invariant under the $\mathbb{Z}_2$ automorphism

$$
\begin{aligned}
Q_- &\leftrightarrow \overline{Q}_-, \\
F_V &\leftrightarrow F_A, \\
Z &\leftrightarrow \tilde{Z},
\end{aligned}
\tag{5}
$$

with all other generators kept intact. This is mirror symmetry on the level of the supersymmetry algebra.

Supersymmetric fields

The two-dimensional $\mathcal{N} = (2, 2)$ fields are representations of the $\mathcal{N} = (2, 2)$ algebra. They are usually defined as functions on the $\mathcal{N} = (2, 2)$ superspace, which is an extension of two-dimensional space-time with four odd directions, parametrised by the fermionic coordinates

$$
\theta^{\pm}, \overline{\theta}^{\pm}. \tag{6}
$$

All θ's are anti-commuting coordinates that are related by complex conjugation,

$$
(\theta^{\pm})^* = \overline{\theta}^{\pm}, \tag{7}
$$

where the $\pm$-index stands for the spin under a Lorentz transformation. Because the fermionic coordinates are anti-commuting, superfields can be Taylor expanded as monomials in the $\theta^{\pm}$ and $\overline{\theta}^{\pm}$.

Some particularly interesting classes of superfields are defined in terms of the supersymmetric covariant derivatives $D_{\pm}$ and $\overline{D}_{\pm}$. The latter are derivatives on $\mathcal{N} = (2, 2)$ superspace that are defined in such a way that

$$
\{D_{\pm}, \overline{D}_{\pm}\} = 2\partial_{\pm}. \tag{8}
$$

We have the:

- chiral superfields Φ, with $\overline{D}_\pm \Phi = 0$ and the expansion

$$\Phi = \phi + \theta^+ \psi_+ + \theta^- \psi_- + \theta^+ \theta^- F, \tag{9}$$

- analogously, anti-chiral superfields $\overline{\Phi}$ with $D_\pm \overline{\Phi} = 0$,
- twisted chiral superfields $\tilde{\Phi}$, with $\overline{D}_+ \tilde{\Phi} = D_- \tilde{\Phi} = 0$ and the expansion

$$\tilde{\Phi} = \tilde{\phi} + \theta^+ \tilde{\psi}_+ + \overline{\theta}^- \tilde{\psi}_- + \theta^+ \overline{\theta}^- G, \tag{10}$$

- and analogously, twisted anti-chiral superfields $\overline{\tilde{\Phi}}$ with $\overline{D}_- \overline{\tilde{\Phi}} = D_+ \overline{\tilde{\Phi}} = 0$.

The two-dimensional field strength can be encoded as an auxiliary term in a (separate) twisted superfield:

$$\Sigma = \sigma + \theta^+ \tilde{\lambda}_+ + \overline{\theta}^- \lambda_- + \theta^+ \overline{\theta}^- (D - iF_{01}). \tag{11}$$

Supersymmetric Lagrangian

The encryption of all fields in superfields, makes it much more simple and elegant to write down Lagrangians. In particular, the $\mathcal{N} = (2,2)$ Lagrangian is simply a sum of Kähler potentials (and their twisted analogues)

$$\int d^2\theta d^2\overline{\theta}\, K(\Phi, \overline{\Phi}) = \int d^2\theta d^2\overline{\theta}\, g_{i\bar{j}} D\Phi^i \overline{D\Phi^j}, \tag{12}$$

and holomorphic superpotentials (and their twisted analogues)

$$\int d^2\theta\, W(\Phi) + c.c. \quad \text{and} \quad \int d^2\overline{\theta}\, \widetilde{W}(\Phi) + c.c, \tag{13}$$

possibly combined with additional kinetic terms for the gauge fields. Expanding this in the components of the superfields yields the expected kinetic and potential terms for the component fields. We will see examples of this later.

(Twisted) chiral ring

Operators $\mathcal{O}$ in the $\mathcal{N} = (2,2)$ theory (which may be thought of as operator-valued products of fields) are called

- chiral if $[\overline{Q}_\pm, \mathcal{O}] = 0$
- anti-chiral if $[Q_\pm, \mathcal{O}] = 0$,
- twisted chiral if $[\overline{Q}_+, \mathcal{O}] = [Q_-, \mathcal{O}] = 0$,
- twisted anti-chiral if $[Q_+, \mathcal{O}] = [\overline{Q}_-, \mathcal{O}] = 0$.

Since half of the supercharges act trivially on these operators, they are half BPS operators.

Similar to the conformal field theory, where one can define a product between two primary operators, the (twisted) chiral operators above form a ring. This ring may be defined in terms of the nilpotent supercharges

$$Q_A^\xi = Q_- + \xi \overline{Q}_+ \quad \text{with} \quad (Q_A^\xi)^2 = 0, \tag{14}$$
$$Q_B^\xi = \overline{Q}_- + \xi \overline{Q}_+ \quad \text{with} \quad (Q_B^\xi)^2 = 0, \tag{15}$$

respectively, for any $\xi \in \mathbb{C}$ with $|\xi| = 1$, and where for a moment we have set $Z = \overline{Z} = 0$. Indeed,

$$Q_A^\xi\text{-closed} \Leftrightarrow \text{twisted chiral}, \tag{16}$$
$$Q_B^\xi\text{-closed} \Leftrightarrow \text{chiral}. \tag{17}$$

To see this, note that the components of Q_A transform with opposite charges under the $U(1)_V$ symmetry, whereas the components of Q_B transforms with opposite charges under the $U(1)_A$ symmetry. Hence, rotating all operators in the equation $[Q_{A,B}, \mathcal{O}] = 0$ with respect to the $U(1)_{V,A}$ symmetry, we conclude that the $Q_{A,B}$-closed operators are equivalent to (twisted) chiral operators. The ring structure defined by the $Q_{A,B}$-cohomology is called the (twisted) chiral ring. It is graded with respect to the $U(1)_{V,A}$-charge.

The proper way to think about the (twisted) chiral ring is in terms of topological twists. Topological twisting is a tool to preserve part of the supersymmetry algebra on a curved (Euclidean) space-time. It works by embedding the Euclidean rotation group in the product of this rotation group and the R-symmetry, and to consider this embedding as the new rotation group.

For the (Euclidean[1]) 2d $\mathcal{N} = (2,2)$ algebra we may choose the new 2d rotation group $U(1)'_E$ to be the diagonal subgroup of the old $U(1)_E \times U(1)_R$, where $U(1)_R$ is either the vector or the axial R-symmetry:

$$\begin{aligned} \text{A-twist}: \quad & U(1)_R = U(1)_V \\ \text{B-twist}: \quad & U(1)_R = U(1)_A. \end{aligned} \tag{18}$$

Since the supercharges $Q_\pm$ and $\overline{Q}_\pm$ have the charges

$$[M_E, Q_\pm] = \pm Q_\pm, \quad [M_E, \overline{Q}_\pm] = \mp \overline{Q}_\pm \tag{19}$$

under the generator M_E of the (old) Euclidean rotation group $U(1)_E$, this implies that the supercharges $Q_{A,\xi}$ and $Q_{B,\xi}$ transform as scalars under the new rotation group $U(1)'_E$, for any choice of ξ. This implies that it makes sense to consider their cohomology. The Q_A-cohomology is also known as the quantum cohomology. (I'll talk a little about this next lecture.)

Note that each supercharge $Q^\xi_{A,B}$, together with its Hermitean conjugate $\overline{Q}^\xi_{A,B}$, generates an $\mathcal{N} = (1,1)$ subalgebra of the $\mathcal{N} = (2,2)$ superalgebra, with

$$\{Q^\xi_A, \overline{Q}^\xi_A\} = -2P + 2i\,\mathrm{Im}(\xi Z), \qquad (Q^\xi_A)^2 = (\overline{Q}^\xi_A)^2 = 0. \tag{20}$$

$$\{Q^\xi_B, \overline{Q}^\xi_B\} = +2H + 2\,\mathrm{Re}(\xi \tilde{Z}), \qquad (Q^\xi_B)^2 = (\overline{Q}^\xi_B)^2 = 0. \tag{21}$$

This subalgebra will appear when we study BPS solitons in the 2d $\mathcal{N} = (2,2)$ theory.[2]

Ground states

Any supersymmetric ground state $|\alpha\rangle$ in the topologically twisted theory can be obtained by acting with the corresponding (twisted) chiral

[1]The $\mathcal{N} = (2,2)$ algebra changes by some factors of i after Wick rotating from Lorentzian to Euclidean time, in such a way that the coordinates σ and τ combine into a complex coordinate $z = \sigma + i\tau$.

[2]In these formulae it is important that that the central charge Z appears in the Q_A-commutator, and $\tilde{Z}$ in the Q_B one. The difference between H and P is less important, I think, since we usually study BPS states in a Euclidean setting in which the coordinates τ and σ are on equal footing.

ring operators on a canonical vacuum state $|0\rangle$. This vacuum state may be obtained by studying the two-dimensional theory on a hemisphere, and stretching it into a long cigar - note here we use the fact that the twisted supercharges are preserved on a curved space. The corresponding boundary state $e^{-tH}|\psi\rangle$ then evolves into the vacuum state denoted by $|0\rangle$. Acting with a (twisted) chiral operator $\mathcal{O}_\alpha$ on $|0\rangle$ produces a new ground state

$$|\alpha\rangle = \mathcal{O}_\alpha|0\rangle. \tag{22}$$

The (twisted) chiral fields and ground states are in 1-1 correspondence whenever the pairing

$$\eta_{\alpha\beta} = \langle\alpha|\beta\rangle, \tag{23}$$

is non-degenerate. This pairing may be visualised by inserting the two chiral operators Q_α, Q_β on either pole of the two-sphere. The pairing is indeed non-degenerate in most examples that we consider (particularly, Landau-Ginzburg models in the B-twist, and linear sigma models in the A-twist). The (twisted) chiral ring structure is then encoded in the equation

$$Q_\alpha Q_\beta = \sum_\gamma C_{\alpha\beta}^{\ \ \gamma}\, Q_\gamma, \tag{24}$$

where $C_{\alpha\beta\eta} = \sum_\gamma C_{\alpha\beta}^{\ \ \gamma}\, \eta_{\gamma\delta}$ computes the three-point function of the (twisted) chiral operators $\mathcal{O}_{\alpha,\beta,\eta}$ on the two-sphere.

1.0.1 Spectral curve

Suppose that we are considering a family T_z of $\mathcal{N} = (2,2)$ theories, related by (twisted) chiral deformations with parameter(s) z. In this case, the (twisted) chiral ring defines a holomorphic bundle of commutative algebras E over the parameter space C. Moreover, because these deformations can be constructed by perturbing the Lagrangian by local operators, we find that there is a holomorphic map of vector bundles

$$q : TC \to E. \tag{25}$$

Dually, we may consider the spectrum Σ_z of the commutative algebra E_z. The point of Σ_z are in 1-1 correspondence with the ground states of the two-dimensional theory T_z. If our two-dimensional theory is a massive theory, i.e. it has a discrete set of ground states and a mass-gap, then the ground states sweep out a branched covering

$$\Sigma \to C. \tag{26}$$

This branched covering has the interpretation of a spectral curve associated to the (possibly higher-dimensional) Higgs bundle (E, φ) over C with Higgs field $\varphi : TC \to \mathrm{End}E$ defined by $(\varphi(v))(w) = q(v) \cdot w$. If the holomorphic map q is an isomorphism, then the spectral curve Σ may be embedded in T^*C.

Surface defects

This all may sound familiar from Andy Neitzke's lectures last week. Remember though that in this two-dimensional setting the space C is the moduli space parametrizing deformations of the 2d theory T_z.

In fact, it turns out these two structures can be reconciled by considering two-dimensional BPS surface defects in the four-dimensional $\mathcal{N} = 2$ theory. In particular, so-called canonical surface defects are such that their moduli space of deformations is equal to the UV curve C, characterizing the 4d $\mathcal{N} = 2$ theory. We will see an example of this later.

Example: Landau-Ginzburg models

Let us illustrate these structures for the illustrative class of Landau-Ginzburg theories. A Landau-Ginzburg model is a 2d $\mathcal{N} = (2, 2)$ theory with n chiral superfields Φ^i and a superpotential $W(\Phi^i)$. The fields Φ^i take values in a Kähler manifold X, whose Kähler metric

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\Phi^i, \overline{\Phi^i}) \tag{27}$$

is defined in terms of the Kähler potential. The superpotential $W : X \to \mathbb{C}$ is a holomorphic function on X. For us, the Kähler manifold X will just be $\mathbb{C}^n$.

Spelling this out in components of the chiral superfields Φ^i, reveals that the corresponding bosonic part in the Lagrangian is given by

$$\mathcal{L}_{\text{bos}} = g_{i\bar{j}}\partial^\mu \phi^i \partial_\mu \overline{\phi^j} + \frac{1}{4} g^{\bar{i}j} \overline{\partial_i W} \partial_j W, \tag{28}$$

after integrating out the auxiliary fields.

The perturbative vacua of the theory are therefore given by the critical points of W, which are the solutions of $\partial_i W = 0$. In the following we assume that the Hessian of W (the matrix of second derivatives) is non-degenerate at every critical point. This implies that the theory is massive, i.e. it has a discrete spectrum and has no massless modes in any vacuum. As a consequence, there is a natural length-scale in the theory.

Suppose the superpotential $W(\phi^i)$ is a holomorphic function of some parameters $z \in C$. The spectral curve Σ is swept out by the critical points of W, whereas the chiral ring is given by the Jacobian ring

$$E_z = \mathbb{C}[\phi^i]/\langle \partial_i W \rangle, \tag{29}$$

where $\langle \partial_i W \rangle$ is the ideal generated by the $\partial_i W$. The map q takes the vector $\partial_z \in TC$ to the element $\partial_z W \in E$.

As a concrete example, suppose that we consider a Landau-Ginzburg model with a single superfield and the superpotential

$$W(\phi) = \frac{1}{3}\phi^3 - z\phi, \tag{30}$$

so that $C = \mathbb{C}_z$. Note that this is a deformation of the quasi-homogeneous superpotential $W(\phi) = \frac{1}{3}\phi^3$ that flows in the IR to the conformal A_2 model.

Since $\partial_\phi W = \phi^2 - z$, the chiral ring is generated by the fields 1 and ϕ with the relation $\phi^2 = z$. The spectral curve Σ is therefore swept out by the equation

$$\Sigma : x^2 = z, \tag{31}$$

where we have introduced $x := \phi$. Note that this is a double covering of $\mathbb{C}$ branched over $z = 0$. The value of the superpotential in the two vacua $x = \pm\sqrt{z}$ of the two-dimensional theory T_z is given by $W(x) = \mp\frac{2}{3}z^{3/2}$.

(picture double covering)

Note that since the superpotential $W(\phi)$ contains all possible chiral deformations – the only chiral operators in this theory are the identity operator and the field ϕ –, the spectral curve Σ is embedded in $T^*\mathbb{C}$.

BPS solitons and MSW complex

You may be worried that there is an issue here. Indeed, remember the (brief) discussion of supersymmetric quantum mechanics (SQM) in Andy's lectures, whose (Euclidean) Lagrangian contains the bosonic terms

$$\mathcal{L}_{\text{bos}}^{\text{SQM}} = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\frac{dh}{dq}^2, \tag{32}$$

in terms of a particle $q(t)$ moving on a Riemannian manifold M and a real Morse function $h : M \to \mathbb{R}$. The SQM supercharge Q is conjugate to the exterior derivative on M and the exact vacuum structure should thus be independent of h.

The resolution in this model is that not all critical points of W are exact vacua. There are non-perturbative contributions, parametrized by BPS solitons that tunnel between the critical points of h, which may lift the vacuum energy. The vacuum structure is governed by the so-called Morse-Smale-Witten complex, whose basis is given by the critical points of h (the perturbative vacua), and whose differential Q counts (with signs) the number of solitons between the perturbative vacua. It's the cohomology of this complex that determines the exact vacua of the SQM.

Since the Landau-Ginzburg model can be reduced to a holomorphic version of supersymmetric quantum mechanics (say by taking all fields to be constant in Euclidean time), a similar story holds here. Let us therefore find out in detail how we find the BPS solitons in the Landau-Ginzburg model.

Suppose that we consider the Landau-Ginzburg model on an interval I_σ times $\mathbb{R}_t$ with coordinates σ and τ. Then we need to specify a boundary condition at the ends of I_σ. Suppose that the fields $\phi^i(\sigma)$ approach the vacuum value ϕ^i_α on one end and ϕ^i_β on the other. With a few manipulations of the Lagrangian (see for instance page 3-4 of [Cecotti-Vafa, hep-th/9211097]), one finds that such a solitonic solution has a minimal energy

$$E_{\alpha\beta} = |W(\beta) - W(\alpha)|, \tag{33}$$

whenever

$$\frac{d\phi^i}{d\sigma} = \frac{\zeta}{2} g^{i\bar{j}} \overline{\partial_j W}, \tag{34}$$

with

$$\zeta = \frac{W(\beta) - W(\alpha)}{|W(\beta) - W(\alpha)|}. \tag{35}$$

Equation (34) is called the ζ-soliton equation. Any solution to this equation corresponds to a BPS soliton with central charge

$$Z_{\alpha\beta} = W(\beta) - W(\alpha). \tag{36}$$

Indeed, since

$$2E_{\alpha\beta} - 2\mathrm{Re}(\zeta^{-1} Z_{\alpha\beta}) = 0, \tag{37}$$

it follows from the Euclidean version of equation (20) that the soliton preserves the Euclidean $\mathcal{N} = (1,1)$ subalgebra generated by Q^ξ_A with $\xi = i\zeta$.

Note that the ζ-soliton equation (34) implies that

$$\partial_\sigma W = \frac{\zeta}{2} g_{i\bar{j}} \partial_i W \overline{\partial_j W} \in \zeta \mathbb{R}_{\geq 0}, \tag{38}$$

so that the quantity

$$\mathbf{H} = \mathrm{Im}(\zeta^{-1} W) \tag{39}$$

is constant along the soliton trajectory. The soliton equation may thus be interpreted as a Hamiltonian flow equation with respect to the Hamiltonian $\mathbf{H}$.

The soliton equation may also be interpreted as a downward flow equation with respect to the Morse function $\mathbf{h} = \mathrm{Re}(\zeta^{-1}W)$. Indeed, the Morse function $\mathbf{h}$ will be decreasing along the flow. (The equivalence of these two forms follows from the Cauchy-Riemann equations for the holomorphic function $\zeta^{-1}W$).

Roughly, the story is now as follows. Fix ζ. For each critical point α of the Morse function $\mathbf{h}$, consider the downward flow with respect to $\mathbf{h}$. This sweeps out so-called Lefschetz thimbles J_α^ζ in the Kähler manifold X. If we rotate ζ it may happen that we encounter critical values $\zeta_{\alpha\beta}$ such that there exist BPS solitons connecting the perturbative vacua labeled by α and β. In that situation, a topology change occurs amongst the Lefschetz thimbles, in which the thimble J_β^ζ stays invariant, but the thimble J_α^ζ picks up a contribution

$$J_\alpha^{\zeta'} = J_\alpha^\zeta + \mu_{\alpha\beta} J_\beta^\zeta, \tag{40}$$

where $\mu_{\alpha\beta}$ is the two-dimensional BPS index that counts (with sign) the number of BPS solitons between the vacua α and β.

(figure with example of Lefschetz thimbles)

Equation (40) might remind you of the Stokes phenomenon, which we will explain later in this section, and leads to a geometric understanding of the Cecotti-Vafa wall-crossing formula.

Example: cubic LG model

Let us consider the cubic Landau-Ginzburg model $T_{z=1}$ as an example, so that $X = \mathbb{C}$ and $W(x) = \frac{1}{3}x^3 - x$. This means that there can be (and in fact are) two solitonic solutions between the vacua at $x_\pm = \pm 1$ with central charge $Z = \mp\frac{2}{3}$ and $\zeta = \mp 1$.

The Lefschetz thimbles $J_\pm^\zeta$ are submanifolds in $X = \mathbb{C}$, containing the critical points $x = \pm 1$, such that the Hamiltonian

$$\mathbf{H}(x) = \mathrm{Im}(\zeta^{-1}W(x)) \tag{41}$$

is constant, while the Morse function

$$\mathbf{h}(x) = \mathrm{Re}(\zeta^{-1}W(x)), \tag{42}$$

is decreasing in the direction of the flow. It is a good exercise to plot the Lefschetz thimbles $J_{\pm}^{\zeta}$ numerically, and check that they are disjoint for generic ζ, but overlap precisely when $\zeta = \pm 1$. Indeed, since $\mathbf{H}(x)$ vanishes along the real axis for $\zeta = \pm 1$, whereas $\mathbf{h}(x)$ decreases/increases along the interval $[-1, 1]$ for $\zeta = \pm 1$, it is clear that the Lefschetz thimbles $J_{+}^{\zeta=\pm 1}$ and $J_{-}^{\zeta=\pm 1}$ overlap on this interval. The interval $[-1, 1] \subset \mathbb{C}$ (with the two possible orientations) thus represents the two BPS solitons with central charges $Z = \pm\frac{2}{3}$.

(add some Mathematica plots)

Spectral networks 1

Let us go back to a generic massive two-dimensional $\mathcal{N} = (2, 2)$ theory T_z. Then we can similarly define BPS solitons (or 2d BPS states) as solutions to the BPS bound

$$Z = \zeta E \tag{43}$$

for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Just as before, these are invariant under the $\mathcal{N} = (1, 1)$ subalgebra generated by Q_A^{ξ} with $\xi = i\zeta$.

Consider the BPS spectrum of the theore T_z as we walk across its deformation space C. The 2d BPS states with central charge $\arg Z = \vartheta$ can be illustrated as paths $\gamma(t)$ on C solving the constraint

$$\mathrm{Im}(e^{-i\vartheta} Z) = 0. \tag{44}$$

Equivalently, such paths may be conveniently plotted by solving the first-order ODE

$$\frac{dZ}{dt} = e^{i\vartheta}, \tag{45}$$

starting at the locus on C where a pair of 2d vacua collides. This was first done in [GMN, WKB], resulting in a beautiful paper with lots of cool pictures. The collection of all such trajectories $\gamma(t)$ for a given phase ϑ was called a spectral network $\mathcal{W}_\vartheta$.

As an example of a spectral network, consider the Landau-Ginzburg model with cubic superpotential

$$W(x) = \frac{1}{3}x^3 - zx, \tag{46}$$

whose two vacua correspond to the two solutions of the equation

$$\Sigma: \; x^2 = z. \tag{47}$$

Fix the phase ϑ. Then there may be a 2d BPS state with central charge such that

$$\arg(Z_{\alpha\beta}) = \vartheta \tag{48}$$

at the locus in C where $W(\beta) - W(\alpha) = \pm\frac{4}{3}z^{3/2}$ has phase ϑ. This gives the following picture on $C = \mathbb{C}$.

(spectral network with three-pronged vertex labeled by 12, 21)

In this picture we have chosen a trivialization of Σ, i.e. a choice of vacuum 1 and 2 across the parameter space C. The orange cross labels the point $z = 0$, where there is only a single vacuum and $Z_{\alpha\beta}$ is thus equal to zero. This point emits three trajectories, which may be oriented in a way that $E_{\alpha\beta}$ increases and labeled by 12 or 21 depending on whether it is the BPS states with central charge Z_{12} or Z_{21} that has phase ϑ along this wall.

How do we interpret this picture? Suppose we fix a point $z \in C$ corresponding to a 2d theory T_z. Then we can vary ϑ and check whether for which values of ϑ there might be walls that run through the point z. If there is such a trajectory with label $\alpha\beta$ for a certain phase $\vartheta_{\alpha\beta}$, then we know that there is a 2d BPS state in the 2d theory T_z connecting the vacua α and β with $\arg Z_{\alpha\beta} = \vartheta_{\alpha\beta}$.

2d wall-crossing

Whereas the 2d BPS spectrum stays invariant under small deformations, as Andy already discussed in his lecture about BPS states, there are real codimension-1 loci on the parameter space C, where

$$Z_{\alpha\beta} + Z_{\beta\gamma} = Z_{\alpha\gamma}. \tag{49}$$

At such a locus the 2d BPS states with central charge $Z_{\alpha\beta}$ and $Z_{\beta\gamma}$ may form a 2d BPS bound state with central charge $Z_{\alpha\gamma}$. These instances of 2d wall-crossing can be conveniently read off from the spectral network $\mathcal{W}_\vartheta$.

Before we explain this, note that to see 2d wall-crossing we need to have at least 3 vacua in the 2d $\mathcal{N} = (2,2)$ theory. Equivalently, the degree of the covering $\Sigma \to C$ should be at least three. The 2d wall-crossing then appears when two trajectories labeled by $\alpha\beta$ and $\beta\gamma$ intersect each other. At such an intersection a new trajectory with label $\alpha\gamma$ may emerge, which corresponds to the new 2d BPS state with label $\alpha\gamma$. In that sense, 2d-wall-crossing is literally the crossing of trajectories, also known as walls, of the spectral network $\mathcal{W}_\vartheta$.

An example is given by the Landau-Ginzburg model with quartic superpotential

$$W(\phi) = \frac{1}{4}\phi^4 - \frac{1}{2}z_1\phi^2 - z_2\phi, \tag{50}$$

whose chiral ring is the Jacobian ring

$$E_z = \mathbb{C}[\phi]/\langle \phi^3 - z_1\phi - z_2 = 0 \rangle, \tag{51}$$

and whose spectral network $\mathcal{W}_\vartheta$ at $\vartheta = \pi/2$ is illustrated below, for $z_1 = -1$ held fixed.

(figure on p18 of BPS-lectures.pdf)

The Cecotti-Vafa wall-crossing formula can be obtained by going around any intersection point of trajectories in a small loop. Suppose we decorate this loop with marked points corresponding to the trajectories of the network, and the intervals between points by the vacuum that may be associated to this interval (the index α that appears in the label of both trajectories on either side of the interval).

(figure of simple 2d wall-crossing, with loop and labels)

To each marked point we may then associate the transformation (40) of Lefschetz thimbles. If a trajectory enters the loop, instead of exiting the loop, we associate to the corresponding marked point the inverse of the transformation (40). Composing all five transformations when going around the loop, and imposing that the resulting transformation is the identity, gives the Cecotti-Vafa wall-crossing formula

$$\begin{aligned} \mu'(\alpha,\beta) &= \mu(\alpha,\beta) \\ \mu'(\beta,\gamma) &= \mu(\beta,\gamma) \\ \mu'(\alpha,\gamma) &= \mu(\alpha,\gamma) \pm \mu(\alpha,\beta)\mu(\beta,\gamma). \end{aligned} \tag{52}$$

The Cecotti-Vafa wall-crossing formula was rederived in this way in [GMN, 2d-4d?].

Note the similarity of this derivation with the derivation of the 4d wall-crossing formula in Andy's lecture. Remember that in that case we considered a so-called BPS scattering diagram embedded in the Coulomb branch $\mathbb{B} \times S^1$. This scattering diagram encodes the 4d BPS states in each vacuum $u \in \mathcal{B}$ with specific $\arg(Z^{\mathrm{4d}}) = \vartheta^{\mathrm{4d}}$. In that case the analog of the chiral ring was an algebra of IR line defects labeled by $\mathcal{X}_\gamma^{\mathrm{IR}}$, obeying the commuting algebra

$$\mathcal{X}_\gamma^{\mathrm{IR}} \mathcal{X}_\mu^{\mathrm{IR}} = (-1)^{\langle \gamma,\mu \rangle} \mathcal{X}_{\gamma+\mu}^{\mathrm{IR}}, \tag{53}$$

that is two line defects, on top of each other, behave the same as a single line defect with the total charge. Remember that these line defects exist for any phase ϑ (they were the line defects preserving the supercharges invariant under the subalgebra labeled by ϑ). In contrast, the 4d BPS states only exist for discrete phases ϑ^{4d}.

Andy argued that the KS wall-crossing automorphism

$$K_\gamma(\mathcal{X}_\mu^{\mathrm{IR}}) = (1 - \mathcal{X}_\gamma^{\mathrm{IR}})^{\langle \gamma,\mu \rangle} \mathcal{X}_\mu^{\mathrm{IR}}, \tag{54}$$

may be explained by considering what happens to the algebra of IR line defects when the phase ϑ is varied across the phase ϑ^{4d} of a four-dimensional BPS state. He mentioned that in this situation the 4d

BPS state sort of gets absorbed by the line defects in a way described by the operation $K_\gamma(\mathcal{X}_\mu^{\mathrm{IR}})$, or better the IR line operator gets dressed. More details can be found in the "Framed BPS states" paper by Gaiotto-Moore-Neitzke [1006.0146].

Spectral networks 2

More generally, for any $\mathcal{N} = (2,2)$ theory whose vacuum structure is encoded in the spectral geometry $\Sigma \subset T^*C$, the central charge Z can be expressed in terms of the tautological 1-form $\lambda = xdz$ restricted to Σ. That is,

$$dZ = \lambda. \tag{55}$$

Suppose that we choose a trivialization of the covering Σ, i.e. a choice of labels α of the vacua across C. Then, the $(\alpha\beta)$-trajectories of the spectral network W_ϑ are then given by all paths $p(t)$ on Σ for which

$$(\lambda_\alpha - \lambda_\beta)(v) \in e^{i\vartheta}\mathbb{R} \tag{56}$$

for any tangent vector v to $p(t)$. The $(\alpha\beta)$-trajectories may then be lifted to open paths $\gamma_{\alpha\beta}$ on Σ connecting the preimages z_α and z_β, such that the tautological 1-form λ has the same phase along the path $\gamma_{\alpha\beta}$.

The tautological 1-form λ, when restricted to Σ, can be expressed in terms of the invariants of the Higgs field φ. In the case that the covering $\Sigma \to C$ is of degree 2, such as for the cubic Landau-Ginzburg model, the trace of φ^2 determines a quadratic differential ϕ_2 on C. We then have

$$\lambda = \sqrt{\phi_2}. \tag{57}$$

The fact that trajectories of the corresponding spectral network do not intersect each other, is geometrically because the spectral network $\mathcal{W}_\vartheta$ is the collection of singular leaves of a foliation of ϕ_2 with phase ϑ. Any degree 2 spectral network thus locally looks like the cubic Landau-Ginzburg network.

Open special Lagrangian discs

So far, we have explained that the 2d BPS states with $\arg(Z_{\alpha\beta}) = \vartheta$ are encoded in the spectral network $\mathcal{W}_\vartheta$ as $(\alpha\beta)$-trajectories, which may be lifted to open paths $\gamma_{\alpha\beta}$ on Σ connecting the preimages z_α and z_β. Note that $\gamma_{\alpha\beta}$ is a boundary component of an open "disc" $D_{\alpha\beta}$. (We put disc in quotation marks because it is really an open cycle that in the simplest situation can be topologically described as an open disc.)

(example open disc)

The BPS condition saying that λ has a constant phase along $\gamma_{\alpha\beta}$ implies that the cycle $D_{\alpha\beta}$ is in fact special Lagrangian with respect to the holomorphic 2-form $e^{-i\vartheta}d\lambda$. That is, the BPS condition in the 2d supersymmetric field theory corresponds to a so-called calibration condition in the associated geometry. (Such a correspondence occurs frequently when studying supersymmetric theories.)

We thus find that the 2d BPS states in the theory T_z can be encoded in the spectral geometry as open "discs" with one boundary component on Σ and one boundary component on the fiber F_z of T^*C at $z \in C$.

Vortices in 2d gauge theory

So far we have illustrated the proporties of 2d $\mathcal{N} = (2,2)$ theories with Landau-Ginzburg examples. In particular, we have not studied gauge theories yet. Let us do something about this.

Considering a 2d $\mathcal{N} = (2,2)$ gauge theory means that the Lagrangian changes in the following way. First of all, we need to add the kinetic term for the gauge field, which can be written as

$$\int d^2\theta d^2\overline{\theta}\, \mathrm{Tr}\, \Sigma^\dagger \Sigma, \tag{58}$$

in terms of the twisted chiral superfield Σ encoding the two-dimensional field strength. Second, we of course need to introduce gauge covariant derivatives to enforce gauge invariance. And last, if the gauge group includes a $U(1)$ factor, it is possible to turn on a twisted superpotential, which is simply of the form

$$\widetilde{W} = t\, \mathrm{Tr}\Sigma, \tag{59}$$

for complex t. The real part r of t is known as the Fayet-Iliopoulos (FI) parameter.

Suppose for instance we consider a 2d $\mathcal{N} = (2,2)$ theory with a $U(k)$ gauge group coupled to n chiral superfields, which transform in the fundamental representation of $U(k)$. The corresponding bosonic part of the Lagrangian reads

$$L_{\rm bos} = \frac{1}{e^2}\left(\frac{1}{2}{\rm Tr}F_A \wedge *F_A + (D_\mu \sigma)^2\right) + \sum_{i=1}^n |D_\mu \phi_i|^2 \qquad (60)$$
$$- \quad \sum_{i=1}^n \phi_i^\dagger \{\sigma, \sigma^\dagger\}\phi_i - \frac{e^2}{4}{\rm Tr}\left(\sum_{i=1}^n \phi_i \phi_i^\dagger - r1_k\right)^2$$

Similar to the way we found the BPS solitons in a LG model, we can now consider this theory in a Euclidean background and write down the energy of a field configuration, with the boundary condition that the fields ϕ_i wind around in a certain way when going the circle around infinity. This winding is characterized by the homotopy group

$$\Pi_1(U(k) \times SU(n)/SU(k)_{\rm diag}) = \mathbb{Z}. \qquad (61)$$

which means that the vortices are labeled by a single winding number $\mathfrak{m}$. (Explain, and check!) Moreover, the scalar field σ we will not play a role here, and we just set it to zero.

The energy of such configurations can then be written in the form

$$E = \int d^2z \, \frac{1}{e^2} \, {\rm Tr}\left(*F_A - \frac{e^2}{2}\left(\sum_{i=1}^n \phi_i \phi_i^\dagger - r1_k\right)\right)^2 \qquad (62)$$
$$+ \int d^2z \sum_{i=1}^n |\overline{D}_A \phi_i|^2 + r \int d^2z \, {\rm Tr} * F_A,$$

(where I have parametrized the Euclidean space-time with complex coordinates z and $\bar{z}$), which shows that the energy

$$E \geq r \, {\rm Tr} \int_{\mathbb{R}^2} F_A \qquad (63)$$

is greater or equal than r times the flux through the surface. This flux is in fact equal to the vortex number $\mathfrak{m}$.

There is an equality for configurations that obey the first order equations

$$*F_A = \frac{e^2}{2}\sum_{i=1}^{N}\phi_i\phi_i^\dagger - r1_k, \quad \overline{D}_A\phi_i = 0. \tag{64}$$

These equations are called the BPS vortex equations. They describe point-like energy configurations, labeled by the flux, or vortex number

$$\mathfrak{m} = c_1(F_A). \tag{65}$$

Note that the vortices are the analogous of point-like instantons in four dimensions. In particular, one can write down a partition function (in the A-twisted theory) that localizes precisely on the vortex equations

$$Z_{\text{vortex}}(z) = \sum_{\mathfrak{m}} z^{\mathfrak{m}} \oint_{\mathcal{M}_{\text{vortex}}(\mathfrak{m})} 1, \tag{66}$$

where $z = e^t$.

2d sigma model

Another way to study this 2d $\mathcal{N} = 2$ gauge theory is to consider its IR description. As usual, we find its moduli space of vacua by minimizing the potential energy

$$U = \frac{e^2}{2}\mathrm{Tr}\left(\sum_{i=1}^{n}\phi_i\phi_i^\dagger - r1_k\right)^2 + \frac{1}{2e^2}\mathrm{Tr}[\sigma,\sigma^\dagger]^2 + \sum_{i=1}^{n}\phi_i^\dagger\{\sigma,\sigma^\dagger\}\phi_i. \tag{67}$$

The moduli space of vacua is then obtained as the quotient

$$\mathcal{M}_{\text{vac}} = \{U = 0\}/U(k). \tag{68}$$

If $r > 0$ and we consider solutions such that scalar σ vanishes, then the vacuum moduli space is parametrized by the Grassmanian of k-planes inside of $\mathbb{C}^n$. (More generally, we could add some chiral fields in the anti-fundamental representation of $U(k)$ to the Lagrangian, and the corresponding moduli space would be the flag manifold.) For example, when $k = 1$ we find that the equation

$$\{\sum_{i=1}^{n} |\phi_i|^2 = r\}/U(1) \tag{69}$$

describes the projective plane $\mathbb{CP}^{n-1}$. This is known as the Higgs branch of the theory (as the gauge group is spontaneously broken and the scalars have dynamically obtained a vev).

In the IR limit, the 2d $\mathcal{N} = 2$ gauge theory may be studied as a non-linear sigma model into the Higgs moduli space. This gives a description of the vortex partition function in terms of quasi-maps from $\mathbb{P}^1$ into the Grassmannian (with suitable boundary conditions at infinity of $\mathbb{P}^1$). This relates the 2d $\mathcal{N} = 2$ theory to Gromov-Witten theory of the Grassmannian.

Another option is to keep the diagonal components of the scalar field σ. Such solutions parametrise the Coulomb branch, where the gauge group is broken to a product of $U(1)$'s. This means that the components of the chiral field ϕ_i have obtained a mass proportional to the eigenvalues of σ^2, which means that to find the proper low energy description, we need to integrate them out.

It is well-known that integrating out the chiral fields introduces an effective twisted superpotential $\tilde{W}^{\text{eff}}$ to the effective description. In the case where $k = 1$ this superpotential is of the form

$$\tilde{W}^{\text{eff}}(\Sigma) = t\Sigma + n\Sigma\,(\log\Sigma - 1). \tag{70}$$

From our previous discussions we know that this means that the twisted vacua are the solutions to the equation

$$\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \sigma} = 0, \tag{71}$$

or in other words, the spectral curve $\tilde{\Sigma}$ is parametrized by the equation

$$\sigma^n = e^t. \tag{72}$$

This equation may also be familiar for you from the perspective of the non-linear sigma model, where it determines the quantum cohomology

$$\mathbb{C}[\sigma]/\langle \sigma^n = e^t \rangle. \tag{73}$$

That is, the deformation of the standard cohomology of $\mathbb{CP}^{n-1}$

In fact, the two descriptions on the Coulomb and Higgs branch are related by mirror symmetry.

Vortex partition function

Computing the partition function

$$Z_{\text{vortex}}(z) = \sum_{\mathfrak{m}} z^{\mathfrak{m}} \oint_{\mathcal{M}_{\text{vortex}}(\mathfrak{m})} 1, \tag{74}$$

, of a 2d $\mathcal{N} = (2, 2)$ theory is not easy in general.

Just like in four dimensions, we need to introduce an additional ingredient to be able to compute the partition function using localisation techniques: the two-dimensional analogue of the Ω-background. Parallel to Nikita's explanation, we can define this two-dimensional Ω-background by starting with a four-dimensional background: a fibration of the two-dimensional spacetime $\mathbb{C}$ over an auxiliary torus T^2. That is, of we go around the circle S^1_w of the torus, we rotate our space-time as

$$z \mapsto \epsilon z \tag{75}$$

and its complex conjugate for going around $S^1_{\overline{w}}$.

More precisely, the metric of the four-dimensional background is given by

$$ds^2 = |dz - iz(\epsilon dw + \overline{\epsilon} d\overline{w})|^2 + |dw|^2, \tag{76}$$

where z is the complex space-time coordinate and w the complex coordinate on the torus. The two-dimensional Ω-background is then obtained by dimensional reduction along the periodic directions. This effectively constrains the vortex dynamics to the origin of the two-dimensional space-time.

Mathematically, this turns out to be equivalent to computing the equivariant volume of the vortex moduli space with respect to the $\mathbb{C}^*$-action $z \mapsto \epsilon z$ on $\mathbb{C}$. With an available quiver description of the vortex moduli space, it is then possible to find explicit expressions.

A simple example is the abelian Higgs model, i.e. the 2d $U(1)$-theory coupled to a single massless chiral multiplet. Remember that it's chiral ring is simply determined by the equation

$$\sigma = e^t. \tag{77}$$

The moduli space of $\mathfrak{m}$ vortices on $\mathbb{C}$ in the abelian Higgs model is simply parametrized by their positions,

$$\mathcal{M}_{\text{vortex}}(\mathfrak{m}) = \mathbb{C}^{\mathfrak{m}}/S_{\mathfrak{m}}, \tag{78}$$

where the quotient by $S_{\mathfrak{m}}$ reflects the fact that the vortices are indistinguishable. The Ω-background just acts as a rotation on each $\mathbb{C}$-factor in the product.

The vortex partition function of the abelian Higgs model can then be computed to just be

$$Z_{\text{vortex}}(z,\epsilon) = \sum_{\mathfrak{m}} z^{\mathfrak{m}} \frac{1}{\epsilon^{\mathfrak{m}}\,\mathfrak{m}!} = \exp\left(\frac{z}{\epsilon}\right), \tag{79}$$

Remember that Nikita explained to us how the four-dimensional Ω-background quantizes the Seiberg-Witten geometry. In two dimensions something similar happens: the two-dimensional Ω-background turns out to quantize the twisted chiral ring equation. Indeed, notice that

$$(\epsilon\partial_t - e^t) Z_{\text{vortex}}(z,\epsilon) = 0 \tag{80}$$

and that the differential operator $\epsilon\partial_t - e^t$ reduces in the limit $\epsilon \to 0$ to the twisted chiral ring equation $\sigma = e^t$,

This simple example is from the paper [Dimofte-Gukov-Hollands]. Many more details about vortex partition functions can for instance be found in [1509.08630, ...]. A few years later much more effective techniques, called supersymmetric localization, were developed that allow one to compute the exact partition function of any $\mathcal{N} = (2,2)$ theory in a supersymmetric background like S^2 (see for instance 1206.2606, 1206.2356]).

Mention relation to gauge-Bethe correspondence.

The Airy function

Another non-gauge example is the cubic Landau-Ginzburg model. In the saddle point approximation its partition function roughly resembles the Airy function

$$Ai(z) = \int d\phi \, \exp\left(\frac{\phi^3}{3} - z\phi\right). \tag{81}$$

Indeed, such statements have been made precise in the framework of topological twists and supersymmetric localization (see for instance [1210.6022]). (This was originally discovered in the context of topological minimal models and the KdV hierarchy. See for instance [9201003, ...].)

In particular, one finds that the 2d partition function in the Ω-background is given by the Airy function

$$Z^{2d}(z, \epsilon) = Ai(z/\epsilon^{2/3}). \tag{82}$$

The chiral ring equation $x^2 = z$ gets deformed into the differential equation

$$(\epsilon^2 \partial_z^2 - z) \, Z^{2d}(z, \epsilon) = 0. \tag{83}$$

Non-perturbative partition function

Yet, we claim that this is not yet the full story. Suppose that we want to compute the partition function of a massive 2d $\mathcal{N} = (2, 2)$ theory in the Ω-background with parameter ϵ. Then we need to pick a boundary condition at infinity. For this we could pick any of the vacua α. We may then compute the resulting two-dimensional partition function with respect to the supersymmetry generators Q_ζ, $\overline{Q}_\zeta$ preserved by the 2d Ω-background. For $\epsilon \in \mathbb{C}$ with fixed $\arg(\epsilon) = \vartheta$, this implies that

$$\zeta = \frac{\epsilon}{|\epsilon|}. \tag{84}$$

We call the resulting partition function

$$Z_\alpha^\vartheta(z, \epsilon), \tag{85}$$

where the phase of ϵ is initially fixed, but we try to analytically continue it to $\epsilon \in \mathbb{C}$.

What happens if we vary the phase ϑ? We know that when ϑ crosses a BPS phase $\vartheta_{\alpha\beta}$, the Lefschetz thimble J_α corresponding to the vacuum α picks up an additional contribution J_β (see equation (40)). This implies that the 2d partition function similarly transforms as

$$Z_\alpha^\vartheta(z,\epsilon) \mapsto Z_\alpha^\vartheta(z,\epsilon) + \mu_{\alpha\beta}\, Z_\beta^\vartheta(z,\epsilon). \tag{86}$$

Hence, the two-dimensional partition function Z^ϑ is locally constant with respect to ϑ and jumps precisely at the phases of 2d BPS particles. Moreover, it turns out that jump of Z^ϑ is proportional to the central charge of the 2d BPS state.

Exact WKB and 2d BPS states

This story fits perfectly into the formalism of exact WKB analysis when C is a complex curve. Details of this analysis are taught this week by Kohei Iwaki, so here we just give a brief summary of how it fits in with our story.

Let us see how this works in an example, say the cubic LG model.

Note 2d BPS states can be interpreted as Lefschetz thimbles.

Other example: $\mathbb{P}^1$-model

Categorification

In the story so far we have picked the simplest boundary conditions possible, labeled by elements of the twisted chiral algebra. General considerations tell us that possible boundary conditions instead form a category, with morphisms defined by pointlike operators that can be inserted at the boundary. In the context of 2d $\mathcal{N} = (2,2)$ theories these categories have been studied in detail by Gaiotto-Moore-Witten.

The way they go about this problem for LG models is very informative. Remember that we considered the theory on a space-like interval times time and found the boundary conditions, labeled by elements α and β in the chiral algebra, by assuming that the fields are constant in time. Instead, GMW consider the two-dimensional theory as a supersymmetric quantum mechanics model with as its target manifold the space of maps from the interval (with ends labeled by elements α and β into the Kähler manifold X. The critical points of this SQM are precisely the solutions to the ζ-soliton equations with boundary conditions given by α and β. So in their picture the BPS-solitons label the boundary conditions as time goes to $\pm\infty$. Since the energy of such a BPS soliton is localised in the center of the interval, we may interpret the soliton as a morphism from the chiral algebra to itself.

(picture rectangle with boundary conditions)

Since the critical points of the SQM are given by the BPS solitons, these form the basis for the "categorified" MSW complex. The differential in this case counts (with signs) the number of solutions to a new ζ-instanton equation, that only preserves a single supersymmetry. It is very interesting research problem to find the corresponding partition functions.

Part 2: 4d QFT's and non-perturbative Seiberg-Witten geometry

Brief review about what we learned about 4d $\mathcal{N}=2$ theories in lectures by Andrew Neitzke and Nikita Nekrasov last week:

Seiberg-Witten geometry

IR geometry for any four-dimensional $\mathcal{N}=2$ theory.
Introduce four-dimensional Seiberg-Witten prepotential. Remember that this can be recovered in the $\epsilon_i \to 0$-limit of the instanton partition function $Z^{\text{inst}}(\epsilon_1, \epsilon_2)$ (which may be computed exactly for a certain class of $\mathcal{N}=2$ theories with a Lagrangian description).

Theories of class S

Write down the "data" describing a class S theory.

4d BPS states

Review what it means to be a four-dimensional BPS state in a Coulomb vacuum u. In particular, note that they are labeled by a phase ϑ (corresponding to their central charge Z) and by an element $\gamma \in \Gamma$ (corresponding to their electro-magnetic charge).

Spectral networks 3

Define spectral networks $\mathcal{W}_{u,\vartheta}$ in terms of the data of class S theories. Argue that 4d BPS states in the vacuum u with central charge $\arg(Z) = \vartheta$ are encoded as saddle trajectories in the spectral network $\mathcal{W}_{u,\vartheta}$. Their electro-magnetic charge γ is the homology class of the 1-cycle on Σ associated to the saddle trajectory.

Discs and M2-branes

Note that the 4d BPS states are realized in terms of special Lagrangian "discs" in T^*C. These correspond to M2-branes ending on the spectral curve in the M5-brane picture. In type IIB they lift to special Lagrangian 3-cycles in the local CY defined by

$$u^2 + v^2 + x^N + q_2(z)x^{N-1} + \ldots + q_N(z) = 0, \tag{87}$$

i.e. the $\mathbb{C}^*$-fibration over T^*C that degenerates precisely over the spectral curve Σ.

Let us now embed in this context what we learned from two-dinmensional $\mathcal{N} = (2, 2)$ theories:

Surface defects

Describe surface defects and how to realise them by coupling a two-dimensional $\mathcal{N} = (2, 2)$ to the four-dimensional $\mathcal{N} = 2$ theory. (This is described in detail in the GMN paper about 2d-4d wall-crossing [1103.2598].)

Canonical surface defects are realised by a family of 2d $\mathcal{N} = (2, 2)$ theories whose parameter space is exactly the UV curve C of the four-dimensional class S theory.

Example: the LG theory can be realised as a canonical BPS surface defect in the AD_n theory.

Example: the $\mathbb{CP}^n$ model can be realised as a canonical surface defect in the pure $SU(n)$ theory.

2d-4d BPS states

The 2d BPS states become so-called 2d-4d BPS states. They obey wall-crossing formulae that are a natural synthesis of the already discussed 2d and 4d wall-crossing formula (see [1103.2598]).

Yet, we want to do much more. Our goal will be to construct a non-perturbative four-dimensional partition function Z_{4d}^{ϑ} that is locally constant in a phase ϑ. To do this, we start by introducing the coordinates $\mathcal{X}_\gamma^\vartheta$ in more detail.

Line defects

Remember the (pure) 4d wall-crossing picture. Here I want to describe a more mathematical understanding of the IR line defects $\mathcal{X}_\gamma^{\text{IR}}$.

Abelianization

Spectral coordinates

Varia

relation to FG and FN coordinates
the variables in wall-crossing formula
relation to exact WKB, quantum periods

Quantum Seiberg-Witten geometry

Ω background
NRS superpotential

Non-perturbative Seiberg-Witten geometry

Example: pure $SU(2)$ theory

Example: $N_f = 4$ $SU(2)$ theory

Holomorphic Floer theory

Spectral problems

Harmonic oscillator example Spectral problems of class S Spectral coordinates and quantization conditions Quantum periods and spectral determinants

Example: Mathieu spectral problems

Example: Heun spectral problems