Les Houches Lectures on
Exact WKB Method and Painlevé Equations

Kohei Iwaki (Univeristy of Tokyo)

Contents

1 Part I : Introduction to Exact WKB Method **2**
1.1 WKB solution, spectral curve and turning points 2
1.1.1 Construction of WKB solution 2
1.1.2 Spectral curve and turning points 4
1.2 Stokes graph and Borel summability 6
1.2.1 Brief review of Borel summation method 6
1.2.2 Stokes graphs 7
1.2.3 Borel summability and sketch of proof 8
1.3 Connection formulas on Stokes curves 10
1.3.1 Voros connection formula around a simple turning point 10
1.3.2 Relation to the path-lifting rule of Gaiotto–Moore–Neitzke 11
1.3.3 Koike connection formula around a simple pole 12
1.4 Description of monodromy/Stokes matrices via Voros periods 13
1.5 Comments and other topics related to Part I 16

2 Part II : Application to Painlevé Equations **17**
2.1 Brief review of Painlevé equations 17
2.1.1 Isomonodromy deformation and τ-function 17
2.1.2 An observation on the spectral curve from the exact WKB perspective 18
2.2 Construction of Painlevé τ-functions by topological recursion 19
2.2.1 Topological recursion for $\Sigma_{P_{\rm I}}$ 19
2.2.2 Quantum curve and its formal monodromy 20
2.2.3 Construction of the τ-function via discrete Fourier transform 21
2.3 From Painlevé equation to resurgent structure in topological recursion 23
2.3.1 Conjectures on Borel summability and connection formulas 23
2.3.2 Computing Stokes multipliers of $(L_{\rm I})$ 25
2.3.3 From isomonodromy to resurgent structure 26
2.4 Comments and other topics related to Part II 27

1 Part I : Introduction to Exact WKB Method

In the first part, we consider the Schödinger-type ODEs of the form

$$\left(\hbar^2 \frac{d^2}{dx^2} - Q(x)\right) \psi(x,\hbar) = 0, \tag{1.1}$$

where $\hbar$ is a parameter and $Q(x)$ is a rational function of x. The goal of Part I is to provide an explanation of the fundamental concepts of the exact WKB method ([92, 69]) and to introduce its applications. In particular, we will see how the monodromy or Stokes matrix of equation (1.1) is described using the Voros periods (also called quantum periods, spectral coordinates) through the exact method. From a geometric perspective, this corresponds to the concept of (non-)abelianization of [38, 39, 47, 49]. This point of view will be useful in studying the Painlevé equation in Part II.

We denote by

$$\phi(x) = Q(x)dx^2 \tag{1.2}$$

the meromorphic quadratic differential on $\mathbb{P}^1$ associated with (1.1). Below we assume the followings.

Assumption 1.1.

- $\hbar$ is a real positive and small parameter; that is, $0 < \hbar \ll 1$.
- ϕ has at least one zero, and all zeros of ϕ are simple.
- ϕ has a pole at $x = \infty$ of order at least 2.

1.1 WKB solution, spectral curve and turning points

Here we recall a construction of (formal) WKB solutions of the Schrödinger-type ODE (1.1). See [69, §2] for details.

1.1.1 Construction of WKB solution

Let us take a new unknown function $P(x,\hbar)$ defined by

$$\psi(x,\hbar) = \exp\left(\int_{x_0}^{x} P(x',\hbar)\, dx'\right) \tag{1.3}$$

with x_0 being a generic base point. Then, it satisfies the so-called Riccati equation

$$\hbar^2 \left(P(x,\hbar)^2 + \frac{dP}{dx}(x,\hbar)\right) = Q(x) \tag{1.4}$$

associated with (1.1). Now we put the following WKB-ansatz:

$$P(x,\hbar) = \sum_{m \geq -1} \hbar^m P_m(x) \tag{1.5}$$

where $P_m(x)$ are assumed to be holomorphic on some domain. The Riccati equation (1.4) then leads the following set of recursion relations for $P_m(x)$, which we call the *WKB recursion*:

$$P_{-1}(x)^2 = Q(x), \tag{1.6}$$

$$2P_{-1}(x)P_0(x) + \frac{dP_{-1}(x)}{dx} = 0, \tag{1.7}$$

$$2P_{-1}(x)P_{m+1}(x) + \sum_{\ell=0}^{m} P_\ell(x)P_{m-\ell}(x) + \frac{dP_m(x)}{dx} = 0 \qquad (m \geq 0). \tag{1.8}$$

We can easily solve the recursion relation term-by-term and obtain two solutions

$$P_{-1}^{(\pm)}(x) = \pm\sqrt{Q(x)}, \quad P_0^{(\pm)}(x) = -\frac{Q'(x)}{4Q(x)}, \quad P_0^{(\pm)}(x) = \pm\frac{4Q(x)Q''(x) - 5(Q'(x))^2}{32Q(x)^{5/2}}, \quad \ldots \tag{1.9}$$

depending on the choice of the square root of (1.6). Thus we obtain two formal solution of Riccati equation (1.4) and the corresponding WKB solutions as

$$P^{(\pm)}(x,\hbar) = \sum_{m\geq -1} \hbar^m P_m^{(\pm)}(x), \quad \psi_\pm(x,\hbar) = \exp\left(\int_{x_0}^{x} P^{(\pm)}(x',\hbar)\, dx'\right). \tag{1.10}$$

Here and in what follows, integrals of formal series are understood as term-wise integrals.

It is convenient to use odd/even decomposition

$$P_{\rm odd}(x,\hbar) = \frac{P^{(+)}(x,\hbar) - P^{(-)}(x,\hbar)}{2}, \quad P_{\rm even}(x,\hbar) = \frac{P^{(+)}(x,\hbar) + P^{(-)}(x,\hbar)}{2}. \tag{1.11}$$

Then, we have $P^{(\pm)}(x,\hbar) = \pm P_{\rm odd}(x,\hbar) + P_{\rm odd}(x,\hbar)$ and

$$P_{\rm even}(x,\hbar) = -\frac{1}{2P_{\rm odd}(x,\hbar)}\frac{dP_{\rm odd}}{dx}(x,\hbar) \tag{1.12}$$

holds as formal series in $\hbar$. Therefore, we can take

$$\psi_\pm(x,\hbar) = \frac{1}{\sqrt{P_{\rm odd}(x,\hbar)}} \exp\left(\pm\int_{x_0}^{x} P_{\rm odd}(x',\hbar)\, dx'\right) \tag{1.13}$$

as the WKB solution, which is differ from (1.10) up to formal series of $\hbar$ with x-independent coefficients. It can also be written as a formal series with an exponential factor:

$$\psi_\pm(x,\hbar) = e^{\pm S(x)/\hbar} \sum_{m\geq 0} \hbar^{m+\frac{1}{2}} \psi_{\pm,m}(x), \quad S(x) = \int_{x_0}^{x} \sqrt{Q(x')}\, dx'. \tag{1.14}$$

If we truncate the series at the first term, then we have the traditional WKB approximation $e^{\pm S(x)/\hbar}Q(x)^{-1/4}$. The WKB solution (1.13) is normalized by specifying the lower endpoint x_0 and the path of integration. We will later discuss how to choose the normalization when describing connection formulas.

Example 1.2 (Airy equation)**.** The Schrödinger-type ODE (1.1) with $Q(x) = x$ is called the *Airy equation*. For this case, we have

$$P_{-1}^{(\pm)}(x) = \sqrt{x}, \quad P_0^{(\pm)}(x) = -\frac{1}{4x}, \quad P_1^{(\pm)}(x) = \mp\frac{5}{32x^{5/2}}, \quad P_2^{(\pm)}(x) = -\frac{15}{64x^4}, \quad \dots \tag{1.15}$$

and

$$\psi_\pm^{\text{Airy}}(x,\hbar) = e^{\pm\frac{2x^{3/2}}{3\hbar}} \frac{\hbar^{1/2}}{x^{1/4}} \left(1 \pm \frac{5}{48x^{3/2}}\hbar + \frac{385}{4608x^3}\hbar^2 + \cdots\right) \tag{1.16}$$

where we have taken the lower endpoint x_0 in (1.13) at ∞ (expect for the leading term). It is known that the rational numbers appearing in the coefficients are related to the intersection numbers on $\mathcal{M}_{g,n}$ through the topological recursion and quantum curves (see [44, 93, 16]).

Remark 1.3. The above construction of WKB solutions can be easily generalized to the case where the Schödinger potential $Q(x)$ has $\hbar$-series expansion $Q = \sum_{m\geq 0} \hbar^m Q_m(x)$, by replacing the right hand side of (1.6), (1.7), (1.8) by $Q_0(x)$, $Q_1(x)$, $Q_{m+2}(x)$, respectively[1]. In the case, the associated quadratic differential (1.2) is replaced by $Q_0(x)dx^2$ (see Exercise 1 below), and we need further conditions to guarantee the Borel summability of WKB solutions in addition to Assumption 1.1 (c.f., [60]).

1.1.2 Spectral curve and turning points

In view of (1.9), it is found that each coefficient of $P^{(\pm)}(x,\hbar)$, the logarithmic derivative of the WKB solution, is multivalued function of x. Therefore, it is natural to consider

Definition 1.4. The Riemann surface

$$\Sigma = \{(x,y) \in \mathbb{C}^2 \mid y^2 = Q(x)\}. \tag{1.17}$$

is called the *(WKB) spectral curve*, or the *classical limit* of the Schrödinger-type ODE (1.1). We also denote by $\overline{\Sigma}$ its compactification.

The Riemann surface Σ is an important geometric object in WKB analysis. We denote by π the projection map $\overline{\Sigma} \ni (x,y) \mapsto x \in \mathbb{P}^1$. We usually identify the point x with a point (x,y) on Σ by taking an appropriate branch cut for the square root $\sqrt{Q(x)}$, and regard x as a local coordinate of Σ as well.

Note that the path of integration in (1.10) or (1.13) should be considered on

$$\Sigma' = \Sigma \setminus \pi^{-1}(\mathrm{Crit}(\phi)), \tag{1.18}$$

where $\mathrm{Crit}(\phi) := \{\text{zeros and poles of } \phi(x)\}$ is the set of critical points of $\phi(x)$. These points also play crucial role in the WKB analysis.

Definition 1.5. A *turning point* of the Schrödinger-type ODE (1.1) is either zero or simple pole[2] of $\phi(x)$. The poles of $\phi(x)$ are called *singular points* of (1.1).

[1] In this case, $P_{\text{odd}}/P_{\text{even}}$ defined in (1.11) can contain even/odd degree terms of $\hbar$.

[2] The traditional WKB analysis only considered the zeros of $Q(x)$ as turning points, but Koike's research ([71]) revealed that simple poles also play a similar role as turning points. Therefore, the above definition is now adopted. To distinguish it from the conventional turning points, it is sometimes referred to as a "turning point of simple pole-type". See also [48].

The WKB solutions cannot be defined not only at singular points but also at turning points. In quantum mechanics, the connection problem of WKB solutions around the turning point has been studied. Such connection formulas will be discussed in §1.3 from the viewpoint of the Stokes phenomenon with respect to $\hbar$.

Before ending this subsection, we note that the WKB solutions thus constructed are usually factorially divergent series of $\hbar$ at any point x.

Proposition 1.6. Let K be an arbitrary compact set that includes neither turning points nor singularities. Then, there exists $C_K, r_K > 0$ such that

$$\sup_{x \in K} |P_m^{(\pm)}(x)| \leq C_K \, r_K^m \, m! \tag{1.19}$$

holds for all $m \geq 0$.

See [2] for a proof of this fact. In next subsection, we will give a criterion to determine when this divergent series can be Borel summable.

Excercise 1. Prove that the Schrödinger-type ODE (1.1) is transformed to

$$\left(\hbar^2 \frac{d^2}{dz^2} - \tilde{Q}(z) \right) \tilde{\psi} = 0, \quad \tilde{Q}(z) = \left(\frac{dx(z)}{dz} \right)^2 Q(x(z)) - \frac{\hbar^2}{2} \{x(z); z\} \tag{1.20}$$

under a holomorphic change of coordinate $x = x(z)$ combined with the gauge transformation $\tilde{\psi} = (dx/dz)^{-1/2} \psi$. Here, $\{x(z); z\}$ is the Schwarzian derivative[3]. Observe that, in the transformation law (1.20), the leading term Q_0 of the potential function precisely satisfies the transformation rule for quadratic differentials. Prove also that the formal series $P_{\rm odd}$ transforms as follows:

$$\tilde{P}_{\rm odd}(z, \hbar) = \frac{dx(z)}{dz} P_{\rm odd}(x(z), \hbar). \tag{1.21}$$

Excercise 2. The Schrödinger-type ODE

$$\left(\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \nu \right) \right) \psi(x, \hbar) = 0 \qquad (\nu \in \mathbb{C}^*) \tag{1.22}$$

is called the *Weber equation*. Verify that $P_m(x) = O(x^{-2})$ holds when $|x| \to +\infty$ if $m \geq 1$ for this equation. Furthermore, compute the integral along the path connecting two pre-images $\infty_\pm \in \overline{\Sigma}$ of $x = \infty$ by π and check that

$$\int_{\infty_-}^{\infty_+} P_m(x) \, dx = \begin{cases} \dfrac{(1 - 2^{1-2k}) B_{2k}}{2k(2k-1)\nu^{2k-1}} & \text{if } m = 2k - 1 \text{ is odd} \\ 0 & \text{if } m = 2k \text{ is even,} \end{cases} \tag{1.23}$$

holds for any $m \geq 1$. Here, B_{2k} is the Bernoulli number defined by

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k \geq 1} \frac{B_{2k}}{(2k)!} t^{2k}. \tag{1.24}$$

[3] Such objects that include the Schwarzian derivative in the transformation law are called *projective connections* (or *sl_2-opers*).

1.2 Stokes graph and Borel summability

To give a rigorous analytic realization of the previously constructed WKB solution, we uses the so-called *Borel summation method* and Écalle's *resurgent analysis*. See [30, 23, 86] for details.

1.2.1 Brief review of Borel summation method

Suppose we have a compact set K such as in Proposition 1.6 and x varies on K. The *Borel transform* of $\psi_\pm$ is defined to be its term-wise inverse Laplace transform[4]

$$\mathcal{B}\psi_\pm(x,\zeta) = \psi_{\pm,B}(x,\zeta) = \sum_{m\geq 0} \frac{\psi_{\pm,m}(x)}{\Gamma(m+\frac{1}{2})}(\zeta \pm S(x))^{m-\frac{1}{2}}, \tag{1.25}$$

which converges thanks to the estimate (1.19). For any fixed x in K, this defines a germ of (multi-valued) function on a punctured neighborhood of the point $\zeta = \mp S(x)$. If $\psi_{\pm,B}(x,\zeta)$ has an analytic continuation with respect to ζ to a domain containing the half line $\mp S(x) + \mathbb{R}_{\geq 0}$, and growths at most exponentially when $|\zeta| \to +\infty$, then the Laplace transform

$$\mathcal{S}\psi_\pm(x,\hbar) = \int_{\mp S(x)}^{\infty} e^{-\zeta/\hbar}\psi_{\pm,B}(x,\zeta)\,d\zeta \tag{1.26}$$

converges for $0 < \hbar \ll 1$. We also require that the convergence of the integral is uniformly with respect to $x \in K$ so that the resulting function (1.26) is also holomorphic in x. We say that the WKB solution $\psi_\pm$ is *Borel summable* (as a series of $\hbar$) on K if the properties mentioned here are satisfied, and call the resulting function (1.26) the *Borel sum* of $\psi_\pm$.

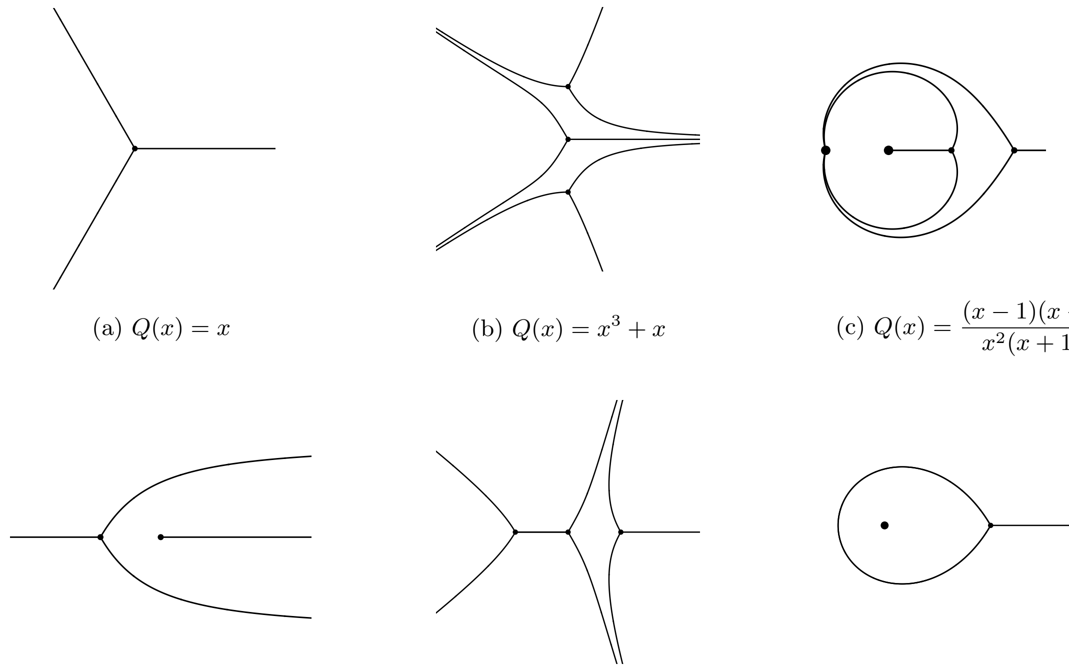
Excercise 3. Prove that the Borel transform $\psi^{\mathrm{Airy}}_{\pm,B}(x,\zeta)$ of the WKB solution of the Airy equation (1.16) can be explicitly written by the Gauss hypergeometric series ${}_2F_1$ as follows:

$$\begin{cases} \psi^{\mathrm{Airy}}_{+,B}(x,\zeta) = \dfrac{\sqrt{3}}{2\sqrt{\pi}}\, x^{-1} t^{-\frac{1}{2}}\, {}_2F_1\Big(\dfrac{1}{6}, \dfrac{5}{6}, \dfrac{1}{2}; t\Big), \\[2ex] \psi^{\mathrm{Airy}}_{-,B}(x,\zeta) = \dfrac{\sqrt{3}}{2\sqrt{\pi}}\, x^{-1} (t-1)^{-\frac{1}{2}}\, {}_2F_1\Big(\dfrac{1}{6}, \dfrac{5}{6}, \dfrac{1}{2}; 1-t\Big), \end{cases} \tag{1.27}$$

where $t = (\zeta + S(x))/2S(x)$. ($S(x) = 2x^{3/2}/3$ for this example). Prove also that $\psi^{\mathrm{Airy}}(x,\hbar)$ are Borel summable if $\operatorname{Im} x^{3/2} \neq 0$ is satisfied.

[4] Here we shift the lower endpoint of the Laplace integral to absorb the exponential factor:

$$e^{\pm S/\hbar}\hbar^{\alpha} = \int_{\mp S}^{\infty} e^{-\zeta/\hbar}\frac{(\zeta \pm S)^{\alpha-1}}{\Gamma(\alpha)}\,d\zeta \qquad (\operatorname{Re}\alpha > 0).$$



(a) $Q(x) = x$ (b) $Q(x) = x^3 + x$ (c) $Q(x) = \dfrac{(x-1)(x-2)}{x^2(x+1)^2}$

(d) $Q(x) = \dfrac{x+1}{x}$ (e) $Q(x) = x^3 - x$ (f) $Q(x) = \dfrac{x-2}{x^2}$

Figure 1.1: Example of Stokes graphs.

1.2.2 Stokes graphs

Since the coefficients of the WKB solution are functions of x, its Borel summability does depend on the position of x. The Stokes graph introduced here provides a criterion for determining the Borel summability of the WKB solution.

Definition 1.7. The *Stokes graph* of the Schrödinger-type ODE (1.1) is a graph on x-plane whose

- vertices are give by zeros and poles of $Q(x)$, and
- edges are given by *Stokes curves*, which are real 1-dimensional curves emanating from a turning point v defined by

$$\mathrm{Im} \int_v^x \sqrt{Q(x')}\, dx' = 0. \tag{1.28}$$

The Stokes curves are nothing but the (horizontal) trajectories of the meromorphic quadratic differential ϕ defined in (1.2). See [87, 19] for properties of these trajectories. The Stokes graph is identical to the *spectral network* for a class S theory specified by ϕ (see [38, 39]).

Figure 1.1 shows some examples of Stokes graphs. The interior of each face of the Stokes graph is called a *Stokes region*. In (e) and (f), we can observe that there exists a Stokes curve connecting (possibly the same) turning points. Such a Stokes curve is called a *saddle connection* (or a *Stokes segment*).

1.2.3 Borel summability and sketch of proof

Now we can give the criterion of the Borel summability of the WKB solutions as follows.

Theorem 1.8. Assume that the Stokes graph of (1.1) does not contain saddle connection. Then, the WKB solution $\psi_{\pm}(x,\hbar)$ defined in (1.13) is Borel summable on each Stokes region. Also, the Borel sum $\mathcal{S}\psi_{\pm}(x,\hbar)$ gives a holomorphic solution of the Schrödinger-type ODE (1.1) on the Stokes region.

Theorem 1.8 was proved by several works including [29, 72, 77, 79]. Below, we will very roughly explain why the trajectory of the quadratic differential $\phi(x)$ controls the Borel summability, following the idea due to Koike–Schäfke. See also [90, §3] for more details.

First, we will investigate the Borel transform of an auxiliary series

$$T(x,\hbar) = \sum_{m\geq 1} \hbar^{m+1} P_m(x) \quad \big(= \hbar\big(P(x,\hbar) - \hbar^{-1}P_{-1}(x) - P_0(x)\big)\big). \tag{1.29}$$

The Riccati equation for P implies

$$2\hbar^{-1}\sqrt{Q(x)}\,T + \frac{dT}{dx} = -\hbar^{-1}T^2 - 2P_0T - \hbar\left(P_0^2 + \frac{dP_0}{dx}\right). \tag{1.30}$$

Now, let us change the local coordinate, known as the Liouville transformation, from x to z defined by

$$z = z(x) = \int_{x_*}^{x} \sqrt{Q(x')}\,dx' \tag{1.31}$$

with a certain reference point x_*, and take the the Borel transform of the both sides. Since the multiplication by $\hbar^{-1}$ is translated into the derivative ∂_ζ for the Borel transform, we have

$$\left(2\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial z}\right) T_B(z,\zeta) = A_1(z)\frac{\partial}{\partial\zeta}T_B * T_B(z,\zeta) + A_2(z)T_B(z,\zeta) + A_3(z), \tag{1.32}$$

where $*$ is the convolution product with respect to ζ, and

$$A_1 = -\frac{1}{\sqrt{Q}}, \quad A_2 = -\frac{2P_0}{\sqrt{Q}}, \quad A_3 = -\frac{1}{\sqrt{Q}}\left(P_0^2 + \frac{dP_0}{dx}\right). \tag{1.33}$$

By performing further computations, we can find the following integro-differential equation for T_B:

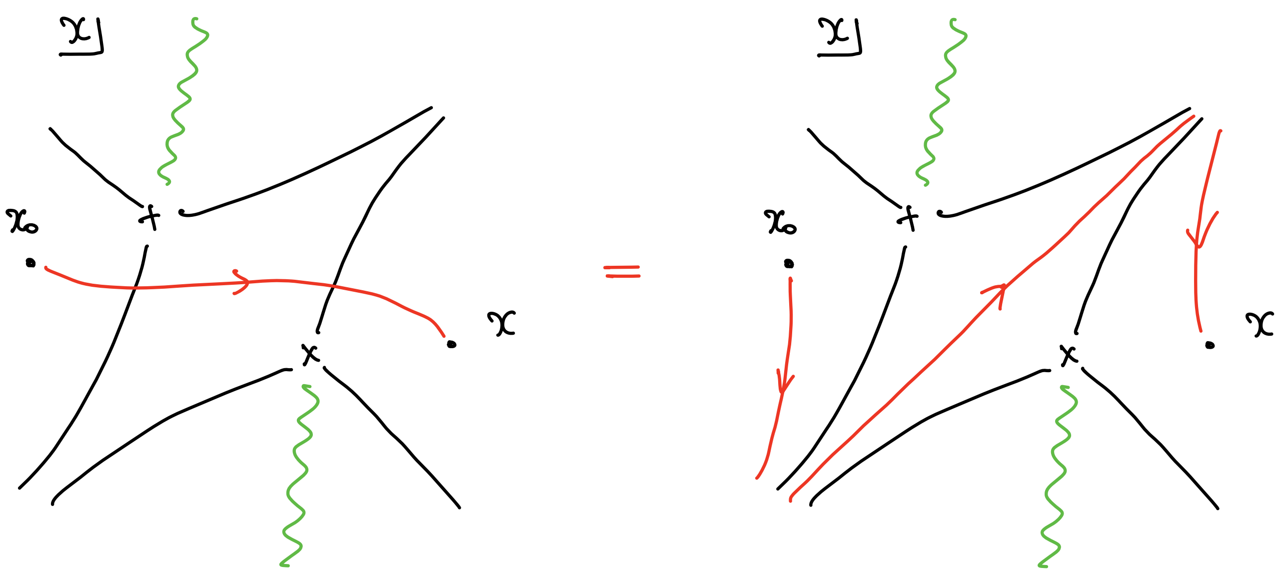
$$\begin{aligned} T_B(z,\zeta) = &\int_0^\zeta A_1\left(z - \frac{\zeta - t}{2}\right)\frac{\partial}{\partial\zeta}T_B * T_B\left(z - \frac{\zeta - t}{2}, t\right)\,dt \\ &+ \int_0^\zeta A_2\left(z - \frac{\zeta - t}{2}\right)T_B\left(z - \frac{\zeta - t}{2}, t\right)\,dt + \int_0^\zeta A_3\left(z - \frac{\zeta - t}{2}\right)\,dt. \end{aligned} \tag{1.34}$$

Koike–Schäfke's approach is to show the existence of analytic continuation of T_B with respect to ζ along the positive real axis by using the equation (1.34). Our T_B is the holomorphic solution near $\zeta = 0$ satisfying

$$T_B(z,0) = 0, \quad \frac{\partial T_B}{\partial\zeta}(z,0) = P_1. \tag{1.35}$$

Now, let us ask if $T_B(z(x_*), \zeta)$ can be analytically continued along the positive real axis on ζ-plane. In this process, the point to be noted is that A_i has singularities at turning points. From the expression $z-(\zeta-t)/2$ included in the equation (1.34), it is clear that to investigate the analyticity with respect to ζ, it is necessary to understand the analyticity with respect to z. As ζ moves in the positive direction along the real axis (with t being satisfying $0 \le t \le \zeta$), $z-(\zeta-t)/2$ moves in the direction where the real part decreases while keeping its imaginary part. Since z is defined by the Liouville transformation equation (1.31), the condition that "the negative part of the trajectory of ψ through x_* does not flow into the turning point" becomes essential in proving the existence of analytic continuation. This is the reason that allows us to determine the Borel summability of the WKB solution using the Stokes graph. After performing these investigations, it is necessary to construct a sequence of successive approximations for the integral equation (1.34) and perform various estimates to prove its convergence. The explanation of these technical aspects will be omitted here.

Once we have constructed the Borel sum of $T(x,\hbar)$ on the complement of the Stokes graph, then one can consider the Borel summability of $\int_{x_0}^{x} T(x',\hbar)\,dx'$. The path of integration can hits several Stokes curves provided if it can be decomposed into a number of paths which are contained in Stokes regions (see the figure below). Then, we can verify that the $\int_{x_0}^{x} T(x',\hbar)\,dx'$ becomes Borel summable since the integrant is Borel summable at any point on the modified path. However, such a deformation is impossible if the original path has non-trivial intersection with saddle connections[5]. This is the reason why the saddle connection gives an obstruction for the Borel summability of WKB solutions (see also Exercise 4 below).



[5]In fact, we can relax the assumption in Theorem 1.8. That is, even if there exist saddle connections, the WKB solution is Borel summable if the path of integration in (1.13) from x_0 to x never intersect with saddle connections.

Excercise 4. Let $W(\hbar)$ be the generating series[6] of the integrals considered in Exercise 2:

$$W(\hbar) = \sum_{k\geq 1} \frac{(1-2^{1-2k})B_{2k}}{2k(2k-1)\,\nu^{2k-1}}\,\hbar^{2k-1} \quad (\nu \in \mathbb{C}^*). \tag{1.36}$$

(1) Compute the Borel transform of W and verify that it is not Borel summable when a saddle connection exists in the Stokes graph of the Weber equation (1.22). (The situation occurs iff $\nu \in i\,\mathbb{R}_{\neq 0}$.)

(2) Show that the Stokes automorphism $\mathfrak{S}$ (c.f., [30, 86]) acts on e^W as follows:

$$\mathfrak{S} e^W = \begin{cases} e^W (1 + e^{2\pi i\nu/\hbar}) & \text{if } \nu \in i\,\mathbb{R}_{>0}, \\ e^W (1 + e^{-2\pi i\nu/\hbar})^{-1} & \text{if } \nu \in i\,\mathbb{R}_{<0}. \end{cases} \tag{1.37}$$

1.3 Connection formulas on Stokes curves

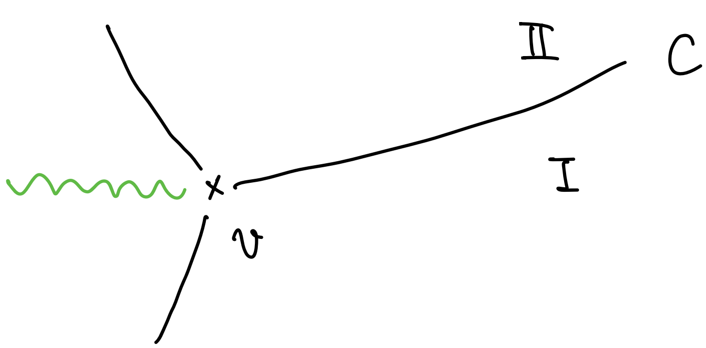
In general, the exact solutions of the Schrödinger-type ODE (1.1) constructed by the Borel summation method in the previous subsection differ across Stokes regions. In this section, we will see the connection formulas that describe how these Borel sums are related.

1.3.1 Voros connection formula around a simple turning point

Here, we will see the connection formulas around the turning points that arise as simple zeros of $\phi(x)$. First, I will describe the setup.

Assumption 1.9.

- Let v is a simple zero of $\phi(x)$, and C be a Stokes curve emanating from v which forms a common boundary of two adjacent Stokes regions I and II. From v, region II appears next to region I in the counter-clockwise direction (see the figure below).
- The Stokes graph of (1.1) does not contain any saddle connection.



[6] This is one of the "Voros period" of the Weber equation (1.22) (for a relative cycle on the spectral curve). We also note the Bernoulli number is related to the Euler characteristic of the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces through topological recursion/quantum curve correspondence. See [45, 83, 55, 56] for more details.

In order to write down the connection formulas, we must fix a normalization of the WKB solution. Here, we adopt the normalization that takes the turning point v at the lower endpoint[7] in the integral in (1.13):

$$\psi_\pm(x,\hbar) = \frac{1}{\sqrt{P_{\rm odd}(x,\hbar)}} \exp\left(\pm \int_v^x P_{\rm odd}(x,\hbar)\, dx \right). \tag{1.38}$$

We also denote by $\Psi_\pm^{\rm J}$ the Borel sum of $\psi_\pm$ defined in the Stokes region ${\rm J} = {\rm I}, {\rm II}$. Then we have the following.

Theorem 1.10 ([92], [69])**.** Under the above setting, the analytic continuation of $\Psi_\pm^{\rm I}$ to the Stokes region II across the Stokes curve C is described as

$$(\Psi_+^{\rm I}, \Psi_-^{\rm I}) = (\Psi_+^{\rm II}, \Psi_-^{\rm II}) \cdot S,$$

$$S = \begin{cases} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} & \text{if } \displaystyle\int_v^x \sqrt{Q(x)}\, dx > 0 \text{ on } C, \\ \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} & \text{if } \displaystyle\int_v^x \sqrt{Q(x)}\, dx < 0 \text{ on } C. \end{cases} \tag{1.39}$$

This is what we call the *Voros connection formula* around a simple turning point. This formula can also be understood as explicitly describing the *Stokes phenomenon* for the divergent series in $\hbar$ (and this is the origin of the name "Stokes graph"). In the case of the Airy equation, this connection formula can be derived by analyzing the Borel singularity; the formula (1.39) follows from several properties of hypergeometric functions (c.f., Exercise 3). The formula for a general Schrödinger-type equation (1.1) is rigorously proven using the "exact WKB-theoretic transformation" to the Airy equation. For details, please refer to [69, Theorem 2.23].

Excercise 5. Show that the Borel sum of the WKB solutions are single-valued around any simple zero v of $Q(x)$.

1.3.2 Relation to the path-lifting rule of Gaiotto–Moore–Neitzke

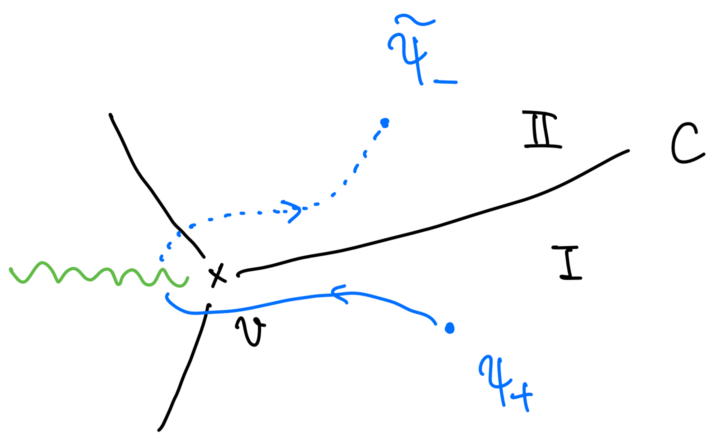
Here we give an alternative description of the Voros connection formula.

Assume that we are in the same situation as the previous subsection, and further assume that $\int_v^x \sqrt{Q(x')}\, dx' > 0$ holds on C. Then, the formula (1.39) for ψ_+ is equivalent to

$$\Psi_+^{\rm I} = \Psi_+^{\rm II} + \widetilde{\Psi}_-^{\rm II}, \tag{1.40}$$

where the second term is the Borel sum of the WKB solution obtained as the term-wise analytic continuation of ψ_+ along the "detoured path" depicted below. Since the path crosses the branch cut and terminates at the different sheet, we put the subscript $-$ for the resulting WKB solution. The description is equivalent to the *path-lifting rule*, which was named in their study of *(non-)abelianization* by Gaiotto–Moore–Neitzke [39].

[7] Although the coefficients of $P_{\rm odd}$ has singularity, the integral with the endpoint v can be defined by means of a contour integration. See [69, §2] for details.



1.3.3 Koike connection formula around a simple pole

It was shown by Koike that a similar connection formula holds on Stokes curves emanating from simple poles ([71]). Below, we will briefly recall this formula. It should be noted that when describing this formula, it is possible to add correction terms in $\hbar$ to the potential function $Q(x)$. This is because, from the perspective of exact WKB analysis, a simple pole is interpreted as a turning point, while it is a regular singular point of (1.1) from the perspective of differential equations. Here, I will present the general formula that allows for the correction term.

Assumption 1.11.

- The potential function Q in (1.1) takes the form

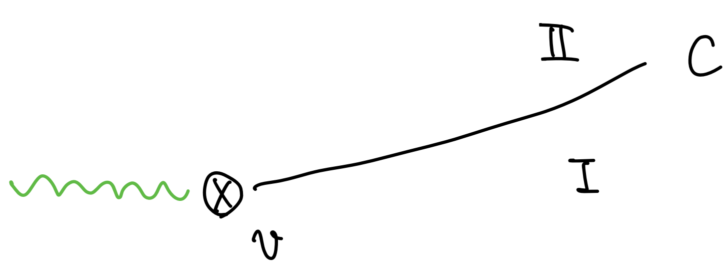
$$Q = Q_0(x) + \hbar^2 Q_2(x), \tag{1.41}$$

  where $Q_0(x)$ has a simple pole at v, while $Q_2(x)$ has at most double pole there.

- Let C be a Stokes curve emanating from v which forms a common boundary of two adjacent Stokes regions I and II. From v, region II appears next to region I in the counter-clockwise direction (see the figure below).

- The Stokes graph of (1.1) does not contain any saddle connection (c.f., Remark 1.3).

As well as the Voros connection formula, we take the WKB solution $\psi_\pm$ normalized at v, which is defined by exactly the same manner as (1.38). Then we have the following.



Theorem 1.12 ([71])**.** Under the above setting, the analytic continuation of $\Psi_{\pm}^{\mathrm{I}}$ to the Stokes region II across the Stokes curve C is described as

$$(\Psi_{+}^{\mathrm{I}}, \Psi_{-}^{\mathrm{I}}) = (\Psi_{+}^{\mathrm{II}}, \Psi_{-}^{\mathrm{II}}) \cdot S,$$

$$S = \begin{cases} \begin{pmatrix} 1 & 0 \\ 2i\cos(\pi\sqrt{1+4\lambda}) & 1 \end{pmatrix} & \text{if } \displaystyle\int_v^x \sqrt{Q(x)}\,dx > 0 \text{ on } C, \\ \begin{pmatrix} 1 & 2i\cos(\pi\sqrt{1+4\lambda}) \\ 0 & 1 \end{pmatrix} & \text{if } \displaystyle\int_v^x \sqrt{Q(x)}\,dx < 0 \text{ on } C, \end{cases} \tag{1.42}$$

where

$$\lambda = \lim_{x\to v}(x-v)^2 Q_2(x). \tag{1.43}$$

Excercise 6. Prove the connection formula (1.42) for the Schrödinger-type ODE (1.1) with the potential

$$Q(x) = \frac{1}{x} + \hbar^2\frac{\lambda}{x^2}. \tag{1.44}$$

(The is a special case of the Bessel equation. We can describe the Borel transform of the WKB solution (1.38) by the Gauss hypergeometric series as well as the Airy equation; c.f., Exercise 3.)

1.4 Description of monodromy/Stokes matrices via Voros periods

The connection formulas discussed in the previous subsection are highly effective for analyzing the global properties of solutions to differential equations. Here, we will briefly explain the content of [69, §3], specifically the calculation of the monodromy matrix/Stokes matrix using the period integrals, which we call the Voros periods, on the spectral curve Σ.

Here, I will introduce the result of calculating the Stokes matrix around the irregular singular point $x = \infty$ for the Weber equation (1.22). This example is quite simple, and the Stokes matrix can be calculated without using the theory of exact WKB analysis. However, from the perspective of understanding how the period integrals arise, it is also an essential example. The monograph [69, §3] addresses more nontrivial examples that truly require the theory of exact WKB analysis, so we recommend referring to it as well.

For simplicity, we consider the case $\nu \in \mathbb{R}_{>0}$. In this case, the Stokes graph of the Weber equation (1.22) is given in Figure 1.2. $x = \infty$ is an irregular singular point of Poincaré rank 2, with four singular directions. These correspond to the directions in which the Stokes curves asymptotically approach. For example, finding the Stokes matrix for the singular direction $\arg x = \pi/2$ is equivalent to determining the connection matrix between the Stokes regions I and III. Let's carry this out.

Here we take the WKB solutions normalized at $x = v_1 = +2\sqrt{\nu}$:

$$\psi_{\pm}(x,\hbar) = \frac{1}{\sqrt{P_{\mathrm{odd}}(x,\hbar)}}\exp\left(\pm\int_{v_1}^x P_{\mathrm{odd}}(x,\hbar)\,dx\right). \tag{1.45}$$

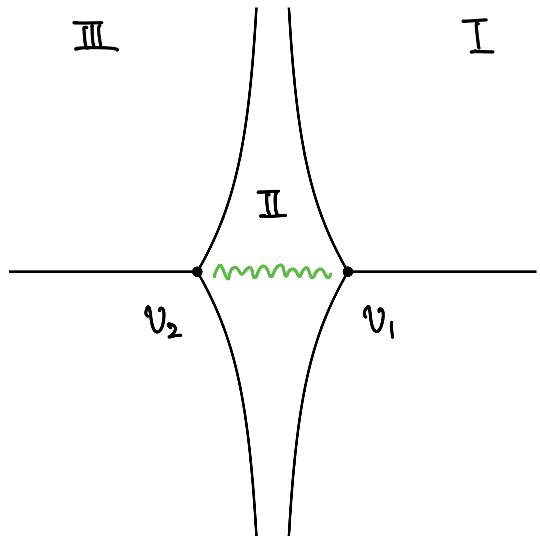


Figure 1.2: The Stokes graph of the Weber equation (1.22).

In computing the analytic continuation from the region I to III passing through II, we will use the Voros formula twice. At the first crossing, we can simply use Theorem 1.10:

$$(\Psi_+^{\mathrm{I}}, \Psi_-^{\mathrm{I}}) = (\Psi_+^{\mathrm{II}}, \Psi_-^{\mathrm{II}}) \cdot \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}. \tag{1.46}$$

In discussing the connection problem at the second crossing, a care must be taken with the normalization of the WKB solution. Note that Theorem 1.10 is valid if the WKB solution is normalized at the turning point. In order to apply Theorem 1.10 at the second crossing point, it is necessary to take the WKB solution

$$\widetilde{\psi}_\pm(x,\hbar) = \frac{1}{\sqrt{P_{\mathrm{odd}}(x,\hbar)}} \exp\left(\pm \int_{v_2}^{x} P_{\mathrm{odd}}(x,\hbar)\, dx \right) \tag{1.47}$$

normalized at the turning point v_2 where the Stokes curve, which we are attempting to cross, emanates. When replacing the lower endpoints of the integration in the WKB solution, the exponential of the period integral emerges as an overall factor as follows:

$$\psi_\pm(x,\hbar) = \exp\left(\frac{1}{2} V_\gamma(\hbar) \right) \widetilde{\psi}_\pm(x,\hbar), \qquad V_\gamma(\hbar) = \oint_\gamma P_{\mathrm{odd}}(x,\hbar)\, dx. \tag{1.48}$$

Here, γ is the closed cycle (which represents a class in $H_1(\Sigma;\mathbb{Z})$) which encircles turning points v_1 and v_2. We note that the series V_γ is also Borel summable if the Stokes graph does not contain any saddle connection (see Footnote 5). Since Theorem 1.10 is valid for the WKB solution $\widetilde{\psi}_\pm$ on the second crossing point, we have the following formula for $\psi_\pm$:

$$(\Psi_+^{\mathrm{II}}, \Psi_-^{\mathrm{II}}) = (\Psi_+^{\mathrm{III}}, \Psi_-^{\mathrm{III}}) \cdot \begin{pmatrix} 1 & ie^{\mathcal{V}_\gamma} \\ 0 & 1 \end{pmatrix}, \tag{1.49}$$

where $\mathcal{V}_\gamma$ is the Borel sum[8] of V_γ. As a conclusion, we have the following resulting connection formula between I to III:

$$(\Psi^{\mathrm{I}}_+, \Psi^{\mathrm{I}}_-) = (\Psi^{\mathrm{III}}_+, \Psi^{\mathrm{III}}_-) \cdot \begin{pmatrix} 1 & i(1 + e^{\mathcal{V}_\gamma}) \\ 0 & 1 \end{pmatrix}. \tag{1.50}$$

Thus we have obtained an explicit expression of the connection matrix for the WKB solution which can be compared with the Stokes matrix of the Weber equation known in literatures.

Remark 1.13. The Stokes matrix of the Weber equation (1.22) computed above is different from the one in literatures. In many references, in computing the Stokes matrix around an irregular singular point, a formal solution expanded at the the irregular singular point is typically used; however, we adopted a different normalization as in (1.45). To compare (1.50) with those results, we should take the WKB solution $\psi_{\pm,\infty}$ normalized at $x = \infty$, and the difference from (1.45) is given by an alternative integral of P_{odd} which was considered already in Exercise 2 and 4:

$$\psi_\pm(x, \hbar) = e^{\frac{1}{2}W(\hbar)} \psi_{\pm,\infty}(x, \hbar). \tag{1.51}$$

The Borel sum of e^W, given by the Γ-function, appears in the expression of the Stokes matrices of $\psi_{\pm,\infty}$, and the formula should be comparable the ones in literatures.

As we have seen, the period integral of $P_{\mathrm{odd}}\, dx$ naturally appears in the expression as a consequence of multiple use of the Voros connection formula. This is also true for more general examples. Those periods are crucially important objects not only in the exact WKB method, but also in the theory of ordinary differential equations on the complex domain.

Definition 1.14. The formal series defined by the period integral[9]

$$V_\gamma(\hbar) = \oint_\gamma P_{\mathrm{odd}}(x, \hbar)\, dx \quad (\gamma \in H_1(\Sigma', \mathbb{Z})) \tag{1.52}$$

is called the *Voros period* for the cycle γ.

In summary, we have

Theorem 1.15 ([69, §3]; see also [85])**.** Suppose that the Stokes graph of the Schrödinger-type ODE (1.1) doesn't contain any saddle connection. Then, the monodromy matrices and Stokes matrices of (1.1) are explicitly described by the Borel sum of the Voros periods.

The spectral curve of the Weber equation (1.22) discussed above is of genus zero. As a result, the Voros period expressed in the aforementioned formula (1.50) can actually be written explicitly as the residue at $x = \infty$ (c.f., Footnote 8). However, in general, when the spectral curve has large genus, the Voros period becomes the generating function of (hyper-)elliptic integrals, and the entries of the monodromy matrix, described as the Borel sum of them, become highly transcendental objects.

---

[8] In this Weber example, V_γ is simply given by $2\pi i\nu/\hbar$ up to sign, so we do not have take the Borel sum.

[9] There are several names for this object. It is also called *quantum period*, *all-order Bohr–Sommerfeld period*, *spectral coordinates*, etc.

Excercise 7. Let us consider the Schrödinger-type ODE (1.1) with an $\hbar$-depending potential

$$Q = Q_0(x) + \hbar Q_1(x) + \hbar^2 Q_2(x) \tag{1.53}$$

and assume that each $Q_i(x)$ has order 2 pole at $x = 0$ (i.e., $x = 0$ is a regular singular point).

(1) For each $m \geq -1$, prove that $P_m^{(\pm)}(x)$ has a pole at $x = 0$ of order at most 1.

(2) Prove that the residues

$$\rho^{(\pm)}(\hbar) = \operatorname*{Res}_{x=0} P^{(\pm)}(x, \hbar)dx \quad \left(= \sum_{m \geq -1} \hbar^m \operatorname*{Res}_{x=0} P_m^{(\pm)}(x)dx \right) \tag{1.54}$$

are convergent series of $\hbar$ and gives the characteristic exponents of (1.1) at $x = 0$ (namely, the eigenvalues of local monodoromy matrix around $x = 0$ is given by $\exp(2\pi i \rho^{(\pm)}(\hbar))$).

1.5 Comments and other topics related to Part I

- The note has dealt only with Schrödinger-type ODEs, but part of the theory of the exact WKB analysis has also been extended to *higher-order ODEs* of the following form:

$$\left(\hbar^n \frac{d^n}{dx^n} + q_1(x)\hbar^{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + q_{n-1}(x)\hbar\frac{d}{dx} + q_n(x) \right) \psi(x, \hbar) = 0. \tag{1.55}$$

  Unlike the second-order case, it has been shown by Berk–Nevins–Roberts ([11]), that discontinuities in the WKB solutions can occur even on *new Stokes curves* that do not originate from turning points. Building on considerations of [11], Aoki–Kawai–Takei proposed a candidate for the Stokes graph of higher-order ODEs [3, 4] (see also [50]). However, a general theorem on the Borel summability of WKB solutions has not yet been established. Giving a rigorous proof of Borel summability and connection formulas, as well as extending the results shown in this note to higher-order ODEs, remain significant and challenging research problems. (Recently, related results were announced in [78].)

- The (candidate of) Stokes graphs of higher-order ODEs were rediscovered in the study of *BPS states* by Gaiotto–Moore–Neiztke in [39], and are also known as *spectral networks*. They also proposed the notion of *(non-)abelianization*; a relation between an $SL_n(\mathbb{C})$-connection on the base x-plane and a $GL_1(\mathbb{C})$-connection on the $n : 1$ covering ([39, 47]). When $n = 2$, this can be understood as the geometric formulation of the result of Theorem 1.15. See [47, 46, 49] for more applications.

- In Exercise 4, we confirmed through explicit calculations that when a saddle connection appears, a discontinuity in the Borel sum, i.e., the Stokes phenomenon, occurs. In fact, similar discontinuities generally happens, as shown in the research in [92, 27] etc., and the associated Voros period satisfies a formula, known as the *Delabaere–Dillinger–Pham (DDP) formula*, that is very similar to (1.37). In fact, the DDP formula has a close relationship with the *(generalized) cluster algebras*, *higher Teichmüller theory* and *BPS structures*. See [38, 39, 19, 60, 61, 18, 7, 6] for more details.

2 Part II : Application to Painlevé Equations

The fact that the exact WKB analysis is effective in describing monodromy naturally leads to exploring its application to Painlevé equations through the theory of isomonodromy deformations of linear ordernary differential equations. In this Part II, we introduce how the general (formal) solutions of the Painlevé equations can be constructed by combining the ideas of the exact WKB method with the theory of topological recursion and quantum curves. Unfortunately, the Borel summability of the constructed formal solutions has not yet been proven. We will propore a conjectural formula on the resurgent structure in the end of this note.

2.1 Brief review of Painlevé equations

Painlevé equations $(P_{\rm J})$ $({\rm J} \in \{{\rm I}, \dots, {\rm VI}\})$ are a class of non-linear ODEs discovered by Painlevé and Gambier in their classification of ODEs with "Painlevé property" which requires the single-valuedness of general solutions away from fixed singular points ([84]; see also [22]). Painlevé equations have commonly nice properties: Hamiltonian description, affine-Weyl symmetry, existence of τ-function, description as isomonodoromy deformation, and connections with mathematical physics.

In this note, we mainly consider the simplest one, called the *first Painlevé equation*[10]:

$$(P_{\rm I}) \ : \ \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t. \tag{2.1}$$

Below, we will review some of aforementioned properties which is relevant to this note.

2.1.1 Isomonodromy deformation and τ-function

All Painlevé equations describe a compatibility condition of a certain system of linear PDEs (c.f., [63]). For the case of $(P_{\rm I})$, the system is given as follows:

$$\begin{cases} (L_{\rm I}) \ : \ \left[\hbar^2 \dfrac{\partial^2}{\partial x^2} - \dfrac{\hbar}{x-q}\left(\hbar\dfrac{\partial}{\partial x} - p\right) - (4x^3 + 2tx + 2H)\right]\psi = 0, \\ (D_{\rm I}) \ : \ \left[\hbar \dfrac{\partial}{\partial t} - \dfrac{1}{2(x-q)}\left(\hbar\dfrac{\partial}{\partial x} - p\right)\right]\psi = 0, \end{cases} \tag{2.2}$$

where

$$H = \frac{p^2}{2} - 2q^3 - tq. \tag{2.3}$$

The compatibility condition of the system (2.2) of PDEs is given by the Hamiltonian system

$$\hbar \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \hbar \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{2.4}$$

[10]The original Painlevé equations do not have $\hbar$-dependence and can be obtained by simply setting $\hbar = 1$. We put the parameter $\hbar$ to apply the exact WKB method. It can be added through a rescaling of variables, and hence, analysis for small $\hbar$ is related to analysis for large t in the original variable. There are many studies on large t asymptotics of Painlevé I; see [37] for example.

which is equivalent to (P_{I}).

(L_{I}) is a linear ODE with irregular singular point at $x = \infty$. The expression (2.3) of H gurantees that $x = q$ is an apparent singularity of (L_{I}); that is, all solution has no monodoromy around the point. Therefore, the Stokes multipliers around $x = \infty$ become the essential monodromy (Stokes) data of (L_{I}). When the system (L_{I}) & (D_{I}) are compatible, the Stokes multipliers of the fundamental solutions remain independent of t, providing the conserved quantities of (P_{I}). In fact, one can show that only two of the Stokes multipliers are independent (see (2.33) below), and they parameterize the general solution of (P_{I}). Such the existence of enough numbers of conserved quantity can be understood as an *integrability* of Painlevé equations. For example, [65, 70, 37] provides a detailed analysis of the behavior of solutions to the Painlevé equations using the t-independence of the monodromy/Stokes datra and the Riemann-Hilbert method.

For a given solution $q(t)$ of (P_{I}), the associated *τ-function* $\tau(t)$ is defined (up to constant) as a function satisfying

$$\hbar^2 \frac{d}{dt} \log \tau(t) = H(t), \tag{2.5}$$

where the right hand side is the corresponding Hamiltonian function (2.3). It is known that $\tau(t)$ is an entire function even though the solution $q(t)$ is meromorphic. The relationship between a solution of (P_{I}) and the τ-function is quite analogous to that between the Weierstrass elliptic function and σ-function (ϑ-function).

Excercise 8.

(1) Derive the equation (2.4) from the compatibility condition of the system (2.2).

(2) Suppose that $a \in \mathbb{C}$ is a pole a solution $q(t)$ of (P_{I}). Compute the first several terms of the Laurent series expansion of $q(t)$ at $t = a$, and observe that the coefficient of $(t-a)^4$ can be chosen as an arbitrary constant. Also, observe that the Hamiltonian function $H(t)$ behaves as

$$H(t) = \frac{\hbar^2}{t-a} \left(1 + O(t-a)\right) \tag{2.6}$$

when $t \to a$, and the corresponding τ-function has a simple zero at a.

2.1.2 An observation on the spectral curve from the exact WKB perspective

As we have seen in the previous subsection, the Stokes multipliers of (L_{I}) become key to analyzing the properties of the general solutions of (P_{I}). When attempting to describe those based on the ideas of the exact WKB analysis, it is expected that the period integrals (Voros periods) on the spectral curve

$$y^2 = 4x^3 + 2tx + 2\text{“}H|_{\hbar=0}\text{”} \tag{2.7}$$

of (L_{I}) will describe those Stokes multipliers.

However, certain issues arise here. Since H is expressed using the solution of (P_{I}), it is necessary to understand the behavior of the solution of (P_{I}) as $\hbar \to 0$ before describing the spectral curve. But, there is no guarantee that the function H has a finite limit as $\hbar \to 0$, making it difficult to start from (2.7).

Let us shift our perspective here. If some meaning could be attached to "$H|_{\hbar=0}$" mentioned above, equation (2.7) would define a family of elliptic curves with the independent variable t of $(P_{\rm I})$ as the deformation parameter. By specifying the A-cycle and B-cycle on these curves, the period integrals of $y\,dx$ along these cycles are expected to describe the two independent Stokes multipliers as the leading term of the Voros periods. Therefore, if the Stokes multiplier of $(L_{\rm I})$ are independent of t, the elliptic curve should be deformed so that the periods of $y\,dx$ remain independent of t. However, since t already appears in the coefficients of equation (2.7), it is impossible to deform the elliptic curve so that all periods of $y\,dx$ are independent of t, which presents a problem with this approach as well.

Therefore, (hoping that the B-cycle can be dealt with later,) let's first consider a deformation family of elliptic curves such that only the period integral of $y\,dx$ along the A-cycle is independent of t:

$$\Sigma_{P_{\rm I}} \ : \ y^2 = 4x^3 + 2tx + u(t,\nu). \tag{2.8}$$

This is a family of elliptic curves obtained by imposing the condition

$$\nu = \frac{1}{2\pi i}\oint_A y\,dx \tag{2.9}$$

with another parameter ν that is independent of t. The equality (2.9) defines $u(t,\nu)$ locally as an implicit function. In fact, this family of elliptic curves is the correct object to consider when constructing the general solution of $(P_{\rm I})$, as will be explained in subsequent sections.

2.2 Construction of Painlevé τ-functions by topological recursion

Based on the idea described in the previous subsection, can we construct a differential equation whose spectral curve is the elliptic curve $\Sigma_{P_{\rm I}}$? If such a construction is possible, then (assuming we can temporarily ignore the issue of t-dependence of the B-period of ydx) it would be close to $(L_{\rm I})$. In fact, it will be shown that this objective can be achieved by the *topological recursion* and *quantum curves*.

2.2.1 Topological recursion for $\Sigma_{P_{\rm I}}$

Following [52], let us we apply the topological recursion to our specific example $\Sigma_{P_{\rm I}}$. For the general formalism of the topological recursion, see [35, 36, 31] for details.

First, we regard (2.8) as an initial data of the topological recursion:

$$C = \mathbb{C}/L, \qquad x = \wp(z), \qquad y = \frac{d\wp}{dz}(z). \tag{2.10}$$

Here, $L = \mathbb{Z}\,\omega_A + \mathbb{Z}\,\omega_B$ is the lattice of periods of dx/y on $\Sigma_{P_{\rm I}}$, and $\wp(z)$ is the associated Weierstrass $\wp$-function. We have already specified the A-cycle and B-cycle in the previous subsection, and the corresponding Bergman bidifferential is given by

$$B(z_1,z_2) = \left(\wp(z_1 - z_2) + \frac{\eta_A}{\omega_A}\right) dz_d z_2. \tag{2.11}$$

The ramification points are given by half periods

$$R = \left\{ \frac{\omega_A}{2}, \frac{\omega_B}{2}, \frac{\omega_A + \omega_B}{2} \right\}, \tag{2.12}$$

and $\sigma(z) \equiv -z \ (\mathrm{mod} L)$ gives the local involution around the ramification point.

Let $W_{g,n}(z_1, \ldots, z_n)$ be the correlators defined by the topological recursion. Namely,

$$W_{0,1}(z) = y(z)\, dx(z), \qquad W_{0,2}(z_1, z_2) = B(z_1, z_2), \tag{2.13}$$

$$W_{g,n+1}(z_0, z_1, \ldots, z_n) = \sum_{r \in R} \operatorname*{Res}_{z=r} \frac{\int_{w=\sigma(z)}^{w=z} W_{0,2}(z_0, w)}{2\big(y(z) - y(\sigma(z))\big) dx(z)} R_{g,n}(z, z_1, \ldots, z_n)$$

$$R_{g,n}(z, z_1, \ldots, z_n) = W_{g-1,n+1}(z, \sigma(z), z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1,\ldots,n\}}}' W_{g_1,1+|I|}(z, z_I)\, W_{g_2,1+|J|}(\sigma(z), z_J), \tag{2.14}$$

where the prime symbol $'$ means that no $W_{0,1}$ appears in the summation. We also define

$$F_g = \frac{1}{2-2g} \sum_{r \in R} \operatorname*{Res}_{z=r} \left(\int^z W_{0,1}(z) \right) W_{g,1}(z) \tag{2.15}$$

for $g \geq 2$. F_0 and F_1 are defined but in an alternative way:

$$F_0 = \frac{tu}{5} + \frac{\nu}{2} \oint_B y dx, \quad F_1 = -\frac{1}{12} \log(\omega_A^6 \mathcal{D}), \tag{2.16}$$

where $\mathcal{D} = -8t^3 - 27u^2$ is the discriminant. See [35] for properties of $\omega_{g,n}$ and F_g.

2.2.2 Quantum curve and its formal monodromy

Let us introduce the following two generating series

$$Z(t, \nu, \hbar) = \exp\left(\sum_{g \geq 0} \hbar^{2g-2} F_g(t, \nu) \right) \tag{2.17}$$

and

$$\begin{aligned} &\chi_{\pm}(x, t, \nu, \hbar) \\ &= \exp\left(\sum_{g \geq 0, n \geq 1} \frac{(\pm\hbar)^{2g-2+n}}{n!} \int_0^{z(x)} \cdots \int_0^{z(x)} \left(W_{g,n}(z_1, \ldots, z_n) - \delta_{g,0}\delta_{n,2} \frac{dx(z_1) dx(z_2)}{(x(z_1) - x(z_2))^2} \right) \right), \end{aligned} \tag{2.18}$$

where $z(x)$ is a local inverse function of $x = \wp(z)$. The first one is called the *perturbative partition function*, while the second one is called the *perturbative wave function*. Both of them are named as "function", but they are divergent series of $\hbar$ in general. (The Borel summability is not rigorosuly proved so far.)

We can show

Theorem 2.1 ([52, Theorem 3.7 and 3.9])**.**

(i) $\chi_\pm$ is a WKB solution of the following PDE:

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - 2\hbar^2 \frac{\partial}{\partial t} - \Big(4x^3 + 2tx + 2\hbar^2 \frac{\partial F}{\partial t}(t,\nu,\hbar)\Big)\right] \chi_\pm(x,t,\nu,\hbar) = 0, \tag{2.19}$$

where $F(t,\nu,\hbar) = \sum_{g\geq 0} \hbar^{2g-2} F_g(t,\nu)$ $(= \log Z(t,\nu,\hbar))$ is the total free energy.

(ii) The term-wise analytic continuation of $\chi_\pm$ along A-cycle and B-cycle are described by

$$\chi_\pm(x,t,\nu,\hbar) \mapsto \begin{cases} e^{\pm 2\pi i \nu/\hbar} \chi_\pm(x,t,\nu,\hbar) & \text{along } A\text{-cycle}, \\ \dfrac{Z(t,\nu\pm\hbar,\hbar)}{Z(t,\nu,\hbar)} \chi_\pm(x,t,\nu\pm\hbar,\hbar) & \text{along } B\text{-cycle}. \end{cases} \tag{2.20}$$

The equation (2.19) is a PDE, and its WKB solutions have not been introduced in this note. However, noting that the $\hbar^2$ is multiplied with the partial derivative with respect to t, it is possible to derive a recursion relation for the coefficients of the WKB-type formal series solution, which is similarly to the method described in §1.1.1. The claim (i) is proved by comparing the recursion relation with the topological recursion. The claim (ii) is a consequence of the *variation formulas* in topological recursion (c.f., [35, §5])

It is not difficult to see that the classical limit of the PDE (2.19) is idential to the spectral curve $\Sigma_{P_{\rm I}}$. Hence, we call the PDE (2.19) the *perturbative quantum curve*. As we expected, the perturbative quantum curve takes a form similar to the isomonodromic linear ODE $(L_{\rm I})$, but they do not exactly match.

Since ν is a parameter independent of t, we were able to construct a differential equation whose formal monodromy (i.e., term-wise analytic continuation) along the A-cycle is t-independent, as initially intended. However, the formal monodromy along the B-cycle is described using difference operators, making it quite complicated. In the next subsection, we will introduce a method to convert such an operator-valued monodromy into a t-independent actual monodromy to obtain an "isomonodoromoic" object.

2.2.3 Construction of the τ-function via discrete Fourier transform

In 2012, Gamayun–Iorgov–Lisovyy constructed the τ-function for the 6-th Painlevé equation $(P_{\rm VI})$ using the *conformal field theory* in [40]. They asserted that the τ-function associated with the general solution of $(P_{\rm VI})$ can be constructed through the *discrete Fourier transform* of the conformal block, and this important formula is now known as the *Kyiv formula*. Meanwhile, Iorgov–Lisovyy–Teschner, in [51], examined the analytic continuation of the conformal block (with the insertion of a degenerate field) and provided a proof of the Kyiv formula. The key fact here was that "the discrete Fourier transform reduces the shift operator-valued monodromy to an actual monodromy"; this provides a clear explanation of why the Kyiv formula of [40] is formulated with the discrete Fourier transform!

In what follows, we will apply this idea to our perturbative wave function[11]. Namely, let us consider the discrete Fourier transform of $\chi_\pm$ with respect to the parameter ν to define

$$\psi_\pm(x,t,\nu,\rho,\hbar) = \frac{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} Z(t,\nu+k\hbar,\hbar)\,\chi_\pm(x,t,\nu+k\hbar,\hbar)}{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} Z(t,\nu+k\hbar,\hbar)}, \tag{2.21}$$

which we call the *non-perturbative wave function* ([34]). Here ρ is a parameter, which is Fourier dual to ν, assumed to be t-independent. Actually, the formal series are not a usual power series in $\hbar$, but can be regarded as a two-sided trans-series (which contain both positive and negative exponential factors). These exponential terms can be summed up to ϑ-functions. The discrete Fourier transform can be regarded as a non-perturbative correction to the perturbative series. See [34, 52] for details.

The previous formal monodromy property (2.20) implies that the term-wise analytic continuation of $\psi_\pm$ becomes

$$\psi_\pm(x,t,\nu,\rho,\hbar) \mapsto \begin{cases} e^{\pm 2\pi i\nu/\hbar}\psi_\pm(x,t,\nu,\rho,\hbar) & \text{along } A\text{-cycle}, \\ e^{\mp 2\pi i\rho/\hbar}\psi_\pm(x,t,\nu,\rho,\hbar) & \text{along } B\text{-cycle}. \end{cases} \tag{2.22}$$

In view of the property (2.22), it is natural to define the *Voros periods of the non-perturbative wave function* along A-cycle and B-cycle by $2\pi i\nu$ and $2\pi i\rho$, respectively, even though the parameter ρ is not an actual period integral on the spectral curve. Based on the idea of the exact WKB method, which states that "the Voros periods are fundamental quantities that describe the monodromy/Stokes data", it is natural to expect that $\psi_\pm$ satisfies an isomonodromy system. In fact, one can prove the following.

Theorem 2.2 ([52, Theorem 4.3 and 4.7])**.** The non-perturbative wave function $\psi_\pm$ given in (2.21) is a formal solution of the isomonodromy system (L_{I}) & (D_{I}) associated with the first Painlevé equation (P_{I}), where q and p in the system (L_{I}) & (D_{I}) are given by

$$q(t,\nu,\rho;\hbar) = -\hbar^2\frac{d^2}{dt^2}\log\tau(t,\nu,\rho,\hbar), \quad p(t,\nu,\rho;\hbar) = -\hbar^3\frac{d^3}{dt^3}\log\tau(t,\nu,\rho,\hbar) \tag{2.23}$$

with

$$\tau(t,\nu,\rho,\hbar) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} Z(t,\nu+k\hbar,\hbar). \tag{2.24}$$

Consequently, (2.24) gives a formal series-valued τ-function for (P_{I}).

The series of the form (2.24) was first introduced by Eynard–Marinõ in [34] as the *non-perturbative partition function*. Theorem 2.2 can be simply summarized as

"non-perturbative quantum curve is the isomonodoromy system".

We also note that it has already been proved by Eynard–Garcia-Failde–Marchal–Orantin that the τ-functions for other all Painlevé equations can be constructed from the topological recursion in the same manner described above ([32, 80, 33]; see also some related results [57, 62, 13, 58]).

[11]By a chain of dualities in theoretical physics, it is expected that our perturbative wave function $\chi_\pm$ is related to the conformal block with a degenerate field insertion. See [74, 9] for some observations about the dualities.

So far, we have not discussed the Borel summability of $\chi_\pm$, Z and the convergence of the Fourier series. Thus, while remaining at the level of formal series, we have constructed a τ-function for (P_{I}) which containing two arbitrary parameters (ν, ρ) by using the topological recursion and discrete Fourier transform, as a topological recursion/exact WKB analogous of the Kyiv formula.

Remark 2.3. Aoki–Kawai–Takei also constructed a class of 2-parameter formal solution of the Painlevé equations in [8, 68, 88], where they used the so-called *multiple-scale analysis.* For example, their 2-parameter solution of (P_{I}) takes the form

$$q(t, \alpha, \beta, \hbar) = q_0(t) + \sum_{\ell \geq 1} \hbar^{\ell/2} q_{\ell/2}(t, \alpha, \beta, \hbar), \tag{2.25}$$

where $q_0(t) = \sqrt{-t/6}$, and the terms $q_{\ell/2}(t, \alpha, \beta, \hbar)$ are functions of t and $\hbar$ containing two free parameters α, β. The first few terms are of the form

$$q_{1/2} = \alpha a_1(t) e^{\varphi(t)/\hbar} + \beta a_{-1}(t) e^{-\varphi(t)/\hbar},$$
$$q_1 = \alpha^2 a_2(t) e^{2\varphi(t)/\hbar} + \alpha\beta a_0(t) + \beta^2 a_{-2}(t) e^{-2\varphi(t)/\hbar}, \ \ldots$$

with a certain functions $\varphi(t)$, $a_i(t)$ (see [69, §4] for their explicit expressions).

This formal solutions were also considered in [41], and studied more extensively in [1, 26, 10, 91] etc. from the viewpoint of resurgence. In particular, the work [10] establishes a relation between our parameters (ν, ρ) and (α, β) appearing above, and hence, the resurgent structure which we will discuss in the next section should be able to compared with those previous results.

2.3 From Painlevé equation to resurgent structure in topological recursion

Here we discuss about an expected resurgent structure of the formal (non-perturbative) series constructed in the previous section. Our strategy is, in studying the resurgent structure of perturbative/non-perturbative partition function, to use the the Stokes multipliers of (L_{I}), which is the conserved quantities for the solution of (P_{I}). In the computation, we will employ the exact WKB method which was reviwed in Part I. A huge difference from Part I is that, since our perturbative quantum curve (2.19) is a PDE, no rigorous theorem about the Borel summability and connection formulas, such as Theorem 1.8, 1.10 and 1.12, has been established.

However, in [52, 59], by assuming that the arguments discussed in Part I are applicable to our quantum curve, we were able to derive an explicit formula for the resurgent structure. Moreover, it turns out that, surprisingly, the formulas obtained by the method perfectly match with results obtained through a completely different approaches. Let us conclude this note by discussing these intriguing observations (with the hope that those will become rigorous mathematical theorems in the near future).

2.3.1 Conjectures on Borel summability and connection formulas

Let us take the meromorphic quadratic differential

$$\phi(x) = (4x^3 + 2tx + u(t, \nu))\, dx^2. \tag{2.26}$$

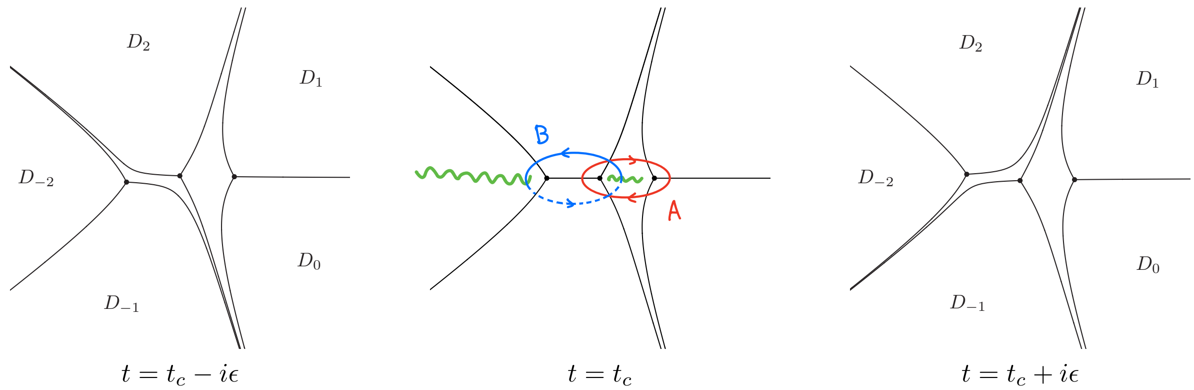


Figure 2.1: Stokes graphs of ϕ when t varies near the negative real axis. In this figure, we have chosen $t_c = -5$, $\epsilon = 1/2$, and $\nu = 1$.

associated with the spectral curve $\Sigma_{P_{\mathrm{I}}}$. Figure 2.1 shows Stokes graphs for some values of t and $u(t,\nu)$, where we have chosen A-cycle and B-cycle as is shown in the middle figure.

The main conjectural ansatz for the following discussion is

Conjecture 2.4 ([52, §5])**.**

(i) If the Stokes graph does not contain any saddle connection, then the perturbative partition function (2.17) is Borel summable. Moreover, the discrete Fourier series

$$\mathscr{T}(t,\nu,\rho,\hbar) = \sum_{k\in\mathbb{Z}} e^{2\pi \mathrm{i} k\rho/\hbar} \mathcal{Z}(t,\nu+k\hbar,\hbar) \tag{2.27}$$

converges and gives an *analytic* τ*-function* of (P_{I}). Here, $\mathcal{Z}$ is the Borel sum of the partition function Z.

(ii) Under the same saddle-free condition, the perturbative wave function $\chi_\pm$ is Borel summable on each Stokes region. Moreover,

$$\Psi_\pm(x,t,\nu,\rho,\hbar) = \frac{\displaystyle\sum_{k\in\mathbb{Z}} e^{2\pi \mathrm{i} k\rho/\hbar}\, \mathcal{Z}(t,\nu+k\hbar,\hbar)\, \mathcal{X}_\pm(x,t,\nu+k\hbar,\hbar)}{\displaystyle\sum_{k\in\mathbb{Z}} e^{2\pi \mathrm{i} k\rho/\hbar}\, \mathcal{Z}(t,\nu+k\hbar,\hbar)} \tag{2.28}$$

converges and give an analytic solution of the isomonodromy system (L_{I}) & (D_{I}) on the Stokes region. Here, $\mathcal{X}_\pm$ is the Borel sum of $\chi_\pm$.

(iii) Under the same saddle-free condition, the Borel sums $\mathcal{X}_\pm$ defined on adjacent Stokes regions are related by the *Voros connection formula* (or the path-lifting rule).

The claim (i) is consistent with the conjectures of [28, 43, 42], while the claims (ii), (iii) are inferred from the results explained in Part I of this note. For a class of genus 0 spectral curves, the conjecture is proved rigorously in [53, 54] where the quantum curve becomes an ODE which can be rigorously handled within the general framework introduced in Part I.

Now, since a saddle connection appears on the B-cycle when t lies on the negative real axis (say $t = -5$ as in Figure 2.1), we may expect:

- The Borel transform of the perturbative partition function has a singularities on the Borel plane at (integer multiples of) the B-periods $\oint_B y dx$.
- The singularity causes a certain Stokes jump when t crosses the negative real axis.

The main goal of the rest of this note is to derive an explicit formula which describes the action of the Stokes automorphism. The resulting formula will be provided in the last subsection, and as a step towards the goal, we will first describe the Stokes multipliers of (L_{I}) which are preserved under the variation of t.

2.3.2 Computing Stokes multipliers of (L_{I})

In what follows, we assume that the claims (i)–(iii) in Conjecture 2.4 are true. Under the assumption (ii), we have five canonical solutions $\Psi_{\pm}^{(j)}$ defined in the Stokes region D_j in Figure 2.1 ($j = 0, \pm 1, \pm 2 \bmod 5$). Then, we define the Stokes matrix S_j attached to the j-th singular direction $\arg x = 2\pi j/5$ by

$$(\Psi_{+}^{(j)}, \Psi_{-}^{(j)}) = (\Psi_{+}^{(j+1)}, \Psi_{-}^{(j+1)}) \cdot S_j. \tag{2.29}$$

We also define the Stokes multiplier s_j as the non-trivial off-diagonal entry of S_j.

The method used in [52, §5] provides us the following results of Stokes multipliers when the Stokes graph is given as in Figure 2.1:

$$\text{At } t = t_c - i\epsilon : \quad \begin{cases} s_{-2} = iX_A \\ s_{-1} = i(X_A^{-1} - X_A^{-1}X_B^{-1} + X_B^{-1}) \\ s_0 \ \ = iX_B \\ s_1 \ \ = i(X_B^{-1} - X_A X_B^{-1}) \\ s_2 \ \ = i(X_A^{-1} - X_A^{-1}X_B), \end{cases} \tag{2.30}$$

$$\text{At } t = t_c + i\epsilon : \quad \begin{cases} s_{-2} = i(X_A - X_A X_B) \\ s_{-1} = i(X_B^{-1} - X_A^{-1}X_B^{-1}) \\ s_0 \ \ = iX_B \\ s_1 \ \ = i(X_A - X_A X_B^{-1} + X_B^{-1}) \\ s_2 \ \ = iX_A^{-1}, \end{cases} \tag{2.31}$$

where X_A and X_B are the exponential Voros periods of our non-perturbative wave function (c.f., (2.22)):

$$X_A = e^{2\pi i \nu/\hbar}, \quad X_B = e^{2\pi i \rho/\hbar}. \tag{2.32}$$

Excercise 9. Assuming Conjecture 2.4 is true, verify that the Stokes multipliers of (L_{I}) are described as above. Also, confirm that the following equation (2.33) holds true.

Although the derivation is based on the conjectural arguments, the resulting Stokes multipliers satisfy the following two desired properties that strongly support the validity of our method:

- The consistency condition

$$1 + s_j s_{j-1} + i s_{j+2} = 0 \qquad (j \bmod 5) \tag{2.33}$$

  which guarantees the single-valuedness of the solutions of $(L_{\rm I})$.

- The elliptic asymptotic formula of the solution of $(P_{\rm I})$:

$$q(t,\nu,\rho,\hbar) = \wp\left(\frac{5t}{4\hbar} + \Big(\frac{\rho}{\hbar}+\frac{1}{2}\Big)\omega_A + \Big(\frac{\nu}{\hbar}+\frac{1}{2}\Big)\omega_B\right) + \text{“}O(\hbar)\text{”}, \tag{2.34}$$

  where the expression can be obtained from an expression of non-perturbative partition function via the ϑ-functions. Kitaev [70] obtain a formula which relates the phase-shift in the elliptic asymptotics with the corresponding Stokes data. We can verify that the Kitaev's formula precisely agrees with our formula (2.30)–(2.31) (see [52, Remark 5.6]).

2.3.3 From isomonodromy to resurgent structure

Now, let us use the results of the previous subsection to derive a Stokes jump formula on the perturbative/non-perturbative partition function. As is mentioned in §2.3.1, we are interested in how the saddle connection on the B-cycle (or the expected Borel singularity $\oint_B y dx$) contributes to the Stokes jump.

Our main claims obtained in [59] are the following:

Conjecture 2.5 ([59])**.** Let $(\nu^\pm, \rho^\pm)$ be two sets of parameters and $\mathscr{T}^\pm(t,\nu^\pm,\rho^\pm,\hbar)$ be the analytic τ-function of $(P_{\rm I})$ defined on a domain that contains the point $t = t_c \pm i\epsilon$, respectively. When t varies and cross the negative real axis, then these analytic τ-functions satisfies the following (non-linear) connection formula:

$$\mathscr{T}^-(t,\nu^-,\rho^-,\hbar) = e^{\frac{1}{2\pi i}{\rm Li}_2(e^{2\pi i\rho^+/\hbar})}\,\mathscr{T}^+(t,\nu^+,\rho^+,\hbar) \tag{2.35}$$

with

$$(\nu^+,\rho^+) = \left(\nu^- - \frac{\hbar}{2\pi i}\log(1-e^{2\pi i\rho^-/\hbar}), \rho^-\right). \tag{2.36}$$

Since t is the isomonodromic time, the Stokes multipliers corresponding to $\mathscr{T}^\pm(t,\nu^\pm,\rho^\pm,\hbar)$ must be identical if they are related by the analytic continuation with respect to t. On the other hand, the Stokes data (2.31) are obtained from (2.30) by the transformation

$$(X_A, X_B) \mapsto (X_A(1-X_B), X_B). \tag{2.37}$$

In fact, this is an example of the *Delabaere–Dillinger–Pham formula* (also known as *cluster transformation*, or *Kontsevich–Soibelman transformation*) which we mentioned in §1.5. The prefactor $\exp(\frac{1}{2\pi i}{\rm Li}_2(e^{2\pi i\rho^+/\hbar}))$ on the right hand side of (2.35) is related to the generating function of the monodromy symplectomorphism (2.37); see [12, 21] for similar results.

We may simply write the formula by means of the Stokes automorphism as

$$\mathfrak{S}\tau(t,\nu,\rho,\hbar) = e^{\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i\rho/\hbar})}\tau\left(t, \nu - \frac{\hbar}{2\pi i}\log(1-e^{2\pi i\rho/\hbar}), \rho, \hbar\right). \tag{2.38}$$

Furthermore, recalling that the τ-function was obtained as the discrete Fourier transform of the perturbative partition function, we can also derive the connection formula for the perturbative partition function by looking at the 0-Fourier mode of the previous formula (2.35)–(2.36). Thus we have

Conjecture 2.6 ([59])**.** The Stokes automorphism associated with the B-periods acts on the perturbative partition function as follows:

$$\mathfrak{S}Z(t,\nu,\hbar) = \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2(e^{-\hbar\partial_\nu}) - \frac{\hbar\partial_\nu}{2\pi i}\log(1-e^{-\hbar\partial_\nu})\right) Z(t,\nu,\hbar). \tag{2.39}$$

The formula (2.39) describes all instanton corrections to the perturbative partition function explicitly in all order:

$$\mathfrak{S}Z(t,\nu,\hbar) = \sum_{n=0}^{\infty} Z^{(n)}(t,\nu,\hbar) \tag{2.40}$$

with

$$Z^{(0)}(t,\nu,\hbar) = Z(t,\nu,\hbar), \quad Z^{(1)}(t,\nu,\hbar) = \left(1 + \frac{\hbar}{2\pi i}\frac{\partial F}{\partial \nu}(t,\nu-\hbar,\hbar)\right) Z(t,\nu-\hbar,\hbar), \quad \dots. \tag{2.41}$$

Note also that the Seiberg–Witten relation $\partial_\nu F_0 = \oint_B y dx$ makes (2.40) a well-defined trans-series. We can verify that the result (2.39) precisely agree with the multi-instanton results for the topological string obtained in [43, 42] (based on the non-perturbative analysis of *holomorphic anomaly equations* initiated by [24, 25]), where the Stokes constant can be identified with the *BPS invarinat*, as expected[12]. We may also observe that the first few terms of the 1-instanton part (2.41) are consistent with a known connection formula for (P_{I}) (see [66] for example). These observations strongly support our heuristic derivation of (2.38).

In summary, the connection formulas (2.35)–(2.36) and (2.39) are a direct consequence of the isomonodoromy property (i.e., *integrability*) of the Painlevé equation and the DDP formula in exact WKB method.

2.4 Comments and other topics related to Part II

- In the latter half of Part II (§2.3), we made numerous considerations that have not been mathematically rigorously proven. For example, proving the Borel summability of the perturbative partition function, analyzing its Borel singularities, and the convergence of the non-perturbative partition function (after taking the Borel), would constitute a significant breakthrough. From a mathematical veiwpoint, in order to state formula (2.39), it is also necessary to prove that the Borel singularity is mild enough to allow the definition of the alien derivative in the first place. Providing a rigorous proof is a

[12] Since the BPS invariant for the B-cycle is equal to 1, it is invisible in the formula (2.39).

challenging task, but the connection with the Painlevé equations might offer some clues for the proof. (For example, methods such as the Riemann-Hilbert approach from [37] might be applicable.)

- The similarity between the construction of τ-functions in Theorem 2.2 and Kyiv formula ([40]) suggests a close relationship among the perturbative partition function of topological recursion, the *conformal blocks* and the *Nekrasov partition functions* (c.f., [76, 5]). As an attempt toward these comparisons, there are [74, 9], but further mathematical research is still desirable. In this direction, recent progress has been made in [14, 20, 15], such as the construction of *Whittaker vectors* based on the *Airy structure* (an algebraic formulation of topological recursion proposed by Kontsevich–Soibelman [73]). Kyiv formula is also generalized partially to the *q-difference Painlevé equations* in [64], so it would be interesting to establish an analogous formula by topological recursion.

Acknowledgement. The author would like to express gratitude to B. Eynard, E. Garcia-Failde, A. Giacchetto, P. Gregori, D. Lewański, D. Mitsios and S. Oukassi who organize the summer school on "Quantum Geometry – Mathematical Methods for Gravity, Gauge Theories and Non-Perturbative Physics" held at Les Houches School of Physics.

References

[1] I. Aniceto, R. Schiappa and M. Vonk, The resurgence of instantons in string theory. *Commub. Number Theor. Phys.*, **6** (2012), 339–496; arXiv:1106.5922 [hep-th].

[2] T. Aoki, T. Kawai, and Y. Takei, The Bender–Wu analysis and the Voros theory, Special Functions: ICM-90 Satellite Conference Proceedings (M. Kashiwara and T. Miwa, eds), Springer–Verlag, 1991, pp. 1–29.

[3] T. Aoki, T. Kawai, Y. Takei, New turning points in the exact WKB analysis for higher-order ordinary differential equations *Analyse Algébrique des Perturbations Singulières, I, Méthodes Résurgentes*, Hermann (1994), pp. 69–84.

[4] T. Aoki, T. Kawai, Y. Takei, On the exact steepest descent method: a new method for the description of Stokes curves *J. Math. Phys.*, **42** (2001), pp. 3691–3713.

[5] L. Alday, D. Gaiotto and Y. Tachikawa, Liouville correlation functions from four-dimensional gauge theories, *Lett. Math. Phys.*, **91** (2010), 167–197; arXiv:0906.3219 [hep-th].

[6] D. Allegretti, Voros symbols as cluster coordinates, *J. Topol.*, **12** (2019), 1031–1068; arXiv:1802.05479 [math.CA].

[7] D. Allegretti and T. Bridgeland, The monodromy of meromorphic projective structures, *Trans. Amer. Math. Soc.*, **373** (2020), 6321–6367; arXiv:1802.02505 [math.GT].

[8] T. Aoki, T. Kawai and Y. Takei, WKB analysis of Painlevé transcendents with a large parameter. II, Structure of Solutions of Differential Equations, World Scientific, 1996, pp.1–49.

[9] H. Awata, H. Fuji, H. Kanno, M. Manabe and Y. Yamada, Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String, *Adv. Theor. Math. Phys.*, **16** (2012), 725–804; ArXiv:1008.0574 [hep-th].

[10] S. Baldino, R. Schiappa, M. Schwick and R. Vega, Resurgent Stokes data for Painlevé equations and two-dimensional quantum (super) gravity, *Commun. Number Theory Phys.*, **17** (2023), 385–552; arXiv:2203.13726.

[11] H. Berk, W.M. Nevins and K. Roberts, New Stokes' line in WKB theory. *Journal of Mathematical Physics*, **23** (1982), 988–1002.

[12] M. Bertola and D. Korotkin, Tau-functions and monodromy symplectomorphisms, *Comm. Math. Phys.*, **388** (2021), 245–290; arXiv:1910.03370 [math.SG].

[13] G. Bonelli, O. Lisovyy, K. Maruyoshi, A. Sciarappa and A. Tanzini, On Painlevé/gauge theory correspondence, *Lett. Math. Phys.*, **107** (2017), 2359–2413; arXiv:1612.06235 [hep-th].

[14] G. Borot, V. Bouchard, N. K. Chidambaram, T. Creutzig, D. Noshchenko, Higher Airy structures, W algebras and topological recursion, ArXiv:1812.08738 [math-ph].

[15] G. Borot, N. K. Chidambaram and G. Umer, Whittaker vectors at finite energy scale, topological recursion and Hurwitz numbers, preprint; arXiv:2403.16938 [math-ph].

[16] V. Bouchard and B. Eyanard, Reconstructing WKB from topological recursion, *Journal de l'Ecole polytechnique – Mathematiques*, **4** (2017), pp. 845–908; arXiv:1606.04498 [math-ph].

[17] P. Boutroux, Recherches sur les transcendentes de M. Painlevé et l'étude asymptotique des équations différentielles du seconde ordre. *Ann. École Norm. Supér.* **30** (1913), 255–375.

[18] T. Bridgeland, Riemann-Hilbert problems from Donaldson-Thomas theory, *Invent. Math.* **216** (2019), 69–124; arXiv:1611.03697 [math.AG].

[19] T. Bridgeland and I. Smith, Quadratic differentials as stability conditions, *Publ. math. IHES.* **121** (2015), 155–278; arXiv:1302.7030 [math.AG].

[20] N.K. Chidambaram, M. Dołęga, and K. Osuga. b-Hurwitz numbers from Whittaker vectors for $\mathcal{W}$-algebras, preprint; arXiv:2401.12814 [math-ph].

[21] I. Coman, P. Longhi and J. Teschner, From quantum curves to topological string partition functions II, preprint; arXiv:2004.04585 [hep-th].

[22] R. Conte and M. Musette, The Painlevé Handbook, Springer Dordrecht, 2008.

[23] O. Costin, Asymptotics and Borel summability, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, Vol. **141**, CRC Press, 2009.

[24] R. Couso-Santamaria, J.D. Edelstein, R. Schiappa and M. Vonk, Resurgent transseries and the holomorphic anomaly: nonperturbative closed strings in local $\mathbb{CP}^2$, *Comm. Math. Phys.*, **338** (2015), 285–346; arXiv:1407.4821 [hep-th].

[25] R. Couso-Santamaria, J.D. Edelstein, R. Schiappa and M. Vonk, Resurgent transseries and the holomorphic anomaly, *Ann. Henri Poincaré*, **17** (2016), 331–399; arXiv:1308.1695 [hep-th].

[26] E. Delabaere, Resurgent methods and the first Painlevé equation, In: *Divergent Series, Summability and Resurgence III.* Lecture Notes in Mathematics, **2155** (2016). Springer, Cham.

[27] E. Delabaere, H. Dillinger and F. Pham, Résurgence de Voros et périodes des courves hyperelliptique, *Annales de l'Institut Fourier*, **43** (1993), 163–199.

[28] N. Drukker, M. Mariño and P. Putrov, Nonperturbative aspects of ABJM theory, *J. High Energy Phys.*, **2011** (2011), no. 11, 141, 29 pages; arXiv:1103.4844.

[29] T. M. Dunster, D. A. Lutz and R. Schäfke, Convergent Liouville-Green expansions for second-order linear differential equations, with an application to Bessel functions, *Proc. Roy. Soc. London*, **A 440** (1993), 37–54.

[30] J. Écalle, Les fonctions résurgentes appliquées à l'itération, Université de Paris-Sud, 1981.

[31] B. Eynard, Counting Surfaces, Progress in Mathematical Physics, vol. **70**, Springer, 2016.

[32] B. Eynard and E. Garcia-Failde, From topological recursion to wave functions and PDEs quantizing hyperelliptic curves, *Forum of Mathematics, Sigma*, **11** (2023), e99; arXiv:1911.07795 [math-ph].

[33] B. Eynard and E. Garcia-Failde, O. Marchal and N. Orantin, Quantization of classical spectral curves via topological recursion, *Commun. Math. Phys.*, **405** (2024); arXiv:2106.04339 [math-ph]

[34] B. Eynard and M. Mariño, A holomorphic and background independent partition function for matrix models and topological strings, *J. Geom. Phys.* **61** (2011), 1181–1202; arXiv:0810.4273 [hep-th].

[35] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion, *Communications in Number Theory and Physics*, **1** (2007), pp. 347–452; arXiv:math-ph/0702045.

[36] B. Eynard and N. Orantin, Topological recursion in enumerative geometry and random matrices, *J. Phys. A: Math. Theor.* **42** (2009), 293001 (117pp).

[37] A.S. Fokas, A.R. Its, A.A. Kapaev and V.Yu. Novokshenov, *Painlevé transcendents: The Riemann-Hilbert approach*, Mathematical Surveys and Monographs, Vol. **128**, Amer. Math. Soc., Providence, RI, 2006.

[38] D. Gaiotto, G.W. Moore and A. Neitzke, Wall-crossing, Hitchin systems, and the WKB approximation, *Adv. in Math.* **234** (2013), 239–403; arXiv:0907.3987 [hep-th].

[39] D. Gaiotto, G.W. Moore and A. Neitzke, Spectral networks, *Ann. Henri Poincaré*, **14** (2013), 1643–1731; arXiv:1204.4824 [hep-th].

[40] O. Gamayun, N. Iorgov and O. Lisovyy, Conformal field theory of Painlevé VI, *JHEP* **10** (2012), 038; arXiv:1207.0787.

[41] S. Garoufalidis, A. Its, A. Kapaev and M. Mariño, Asymptotics of the instantons of Painlevé I, *Int. Math. Res. Not.*, **2012** (2012), 561–606; arXiv:1002.3634 [math.CA].

[42] J. Gu, A. Kashani-Poor, A. Klemm and M. Mariño, Non-perturbative topological string theory on compact Calabi-Yau 3-folds, *SciPost Phys.*, **16** (2024), 079; arXiv:2305.19916 [hep-th].

[43] J. Gu and M. Mariño, Exact multi-instantons in topological string theory, *SciPost Phys.* **15** (2023), 179, 36pages; arXiv:2211.01403 [hep-th].

[44] S. Gukov and P. Sułkowski, A-polynomial, B-model, and quantization, *JHEP*, **2012** (2012), 70.

[45] J. Harer and D. Zagier, The Euler characteristic of the moduli space of curves, *Invent. Math.*, **85** (1986), 457–485.

[46] L. Hollands and O. Kidwai, Higher length-twist coordinates, generalized Heun's opers, and twisted superpotentials, *Adv. Theor. Math. Phys.*, **22** (2018), 1713–1822; arXiv:1710.04438 [hep-th].

[47] L. Hollands and A. Neitzke, Spectral networks and Fenchel–Nielsen coordinates, *Letters in Mathematical Physics*, **106** (2016); arXiv:1312.2979 [math.GT].

[48] L. Hollands and A. Neitzke, BPS states in the Minahan–Nemeschansky E_6 theory, *Comm. Math. Phys.*, **353** (2017), 317–351: arXiv:1607.01743 [hep-th].

[49] L. Hollands and A. Neitzke, Exact WKB and abelianization for the T_3 equation, *Commun. Math. Phys.*, **380** (2020), 131–186; arXiv:1906.04271 [hep-th].

[50] N. Honda, T. Kawai and Y. Takei, Virtual Turning Points, SpringerBriefs in Mathematical Physics, Springer, Tokyo (2015).

[51] N. Iorgov, O. Lisovyy and J. Teschner, Isomonodromic tau-functions from Liouville conformal blocks, *Commun. Math. Phys.*, **336**, (2015), 671-694; arXiv:1401.6104 [hep-th].

[52] K. Iwaki, 2-Parameter τ-Function for the First Painlevé Equation: Topological Recursion and Direct Monodromy Problem via Exact WKB Analysis, *Commun. Math. Phys.*, **377** (2020), 1047–1098; arXiv:1902.06439 [math-ph].

[53] K. Iwaki and O. Kidwai, Topological recursion and uncoupled BPS structures I: BPS spectrum and free energies, *Adv. Math.*, **398** (2022), 108191, 54 pages; arXiv:2010.05596 [math-ph].

[54] K. Iwaki and O. Kidwai, Topological recursion and uncoupled BPS structures II: Voros symbols and the τ-function, *Comm. Math. Phys.*, **399** (2023), 519–572; arXiv:2108.06995 [math-ph].

[55] K. Iwaki, T. Koike and Y.-M. Takei, Voros Coefficients for the Hypergeometric Differential Equations and Eynard-Orantin's Topological Recursion - Part I : For the Weber Equation, *Ann. Henri Poincaré*, **24**, 1305–1353 (2023); arXiv:1805.10945.

[56] K. Iwaki, T. Koike and Y.-M. Takei, Voros Coefficients for the Hypergeometric Differential Equations and Eynard-Orantin's Topological Recursion - Part II : For the Confluent Family of Hypergeometric Equations, *Journal of Integrable Systems*, **4** (2019), xyz004; arXiv:1810.02946 [math.CA].

[57] K. Iwaki and O. Marchal, Painlevé 2 equation with arbitrary monodromy parameter, topological recursion and determinantal formulas, *Ann. Henri Poincaré*, **18** (2017), 2581–2620; arXiv:1411.0875.

[58] K. Iwaki, O. Marchal and A. Saenz, Painlevé equations, topological type property and reconstruction by the topological recursion, *J. Geom. Phys.*, **124** (2018), 16–54; arXiv:1601.02517.

[59] K. Iwaki and M. Mariño, Resurgent Structure of the Topological String and the First Painlevé Equation, *SIGMA*, **20** (2024), 028, 21 pages; arXiv:2307.02080 [hep-th].

[60] K. Iwaki and T. Nakanishi, Exact WKB analysis and cluster algebras, *J. Phys. A: Math. Theor.* **47** (2014), 474009; arXiv:1401.7094 [math.CA].

[61] K. Iwaki and T. Nakanishi, Exact WKB analysis and cluster algebras II: simple poles, orbifold points, and generalized cluster algebras, *Int. Math. Res. Not.*, **14** (2016), 4375–4417; arXiv:1409.4641 [math.CA].

[62] K. Iwaki and A. Saenz, Quantum Curve and the First Painlevé Equation, *SIGMA.,* (**12**), 2016; arXiv:1507.06557.

[63] M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function, *Phys. D* **2** (1981), 306–352.

[64] M. Jimbo, H. Nagoya and H. Sakai, CFT approach to the q-Painlevé VI equation, *Journal of Integrable Systems*, **2** (2017), xyx009; arXiv:1706.01940 [math-ph].

[65] A. Kapaev, Asymptotics of solutions of the Painlevé equation of the first kind, Diff. Uravnenija, **24** (1988), 1684–1695. (English Transl.: Differ. Equ., **24** (1989), 1107–1115.)

[66] A. Kapaev, Quasi-linear stokes phenomenon for the Painlevé first equation, *J. Phys. A: Math. Theor.*, **37** (2004), 11149–11167; arXiv:nlin.SI/0404026.

[67] T. Kawai and Y. Takei, WKB analysis of Painlevé transcendents with a large parameter. I, *Adv. Math.* **118** (1996), 1–33.

[68] T. Kawai and Y. Takei, WKB analysis of Painlevé transcendents with a large parameter. III, *Adv. Math.*, **134** (1998), 178–218.

[69] 河合隆裕, 竹井義次, 特異摂動の代数解析学, 岩波書店, 1998. (English version: Kawai, T. and Takei. Y., *Algebraic Analysis of Singular Perturbation Theory*, American Mathematical Society, Translations of Mathematical Monographs, vol. 227. 2005.)

[70] A.V. Kitaev, Elliptic asymptotics of the first and second Painlevé transcendents, *Russian Math. Surveys*, **49** (1994), 81–150.

[71] T. Koike, On the exact WKB analysis of second order linear ordinary differential equations with simple poles, *Publ. RIMS*, Kyoto Univ., **36** (2000), 297 – 319.

[72] T. Koike and R. Schäfke, On the Borel summability of WKB solutions of Schrödinger equations with rational potentials and its application, in preparation; also Talk given by T. Koike in the RIMS workshop "Exact WKB analysis — Borel summability of WKB solutions", September, 2010.

[73] M. Kontsevich and Y. Soibelman, Airy structures and symplectic geometry of topological recursion, Proceedings of Symposia in Pure Mathematics, **100**, 2018. arXiv:1701.09137.

[74] C. Kozçaz, S. Pasquetti and N. Wyllard, A & B model approaches to surface operators and Toda theories, *J. High Energ. Phys.*, **2010**, 42 (2010); arXiv:1004.2025 [hep-th].

[75] O. Lisovyy and J. Roussillon, On the connection problem for Painlevé I, *J. Phys. A: Math. Theor.*, **50** (2017), 255202; arXiv:1612.08382 [nlin.SI].

[76] N. A. Nekrasov, Seiberg-Witten Prepotential From Instanton Counting, *Adv. Theor. Math. Phys.*, **7** (2004), 831–864; arXiv:hep-th/0206161.

[77] G. Nemes, On the Borel summability of WKB solutions of certain Schrödinger-type differential equations, *Journal of Approximation Theory*, **265** (2021), Article 105562; arXiv:2004.13367 [math.CA].

[78] G. Nemes, On the Borel summability of formal solutions of certain higher-order linear ordinary differential equations, preprint; arXiv:2312.14449 [math.CA].

[79] N. Nikolaev, Existence and Uniqueness of Exact WKB Solutions for Second-Order Singularly Perturbed Linear ODEs, *Commun. Math. Phys.*, **400** (2023), pp. 463–517; arXiv:2106.10248 [math.AP].

[80] O. Marchal and N. Orantin, Quantization of hyper-elliptic curves from isomonodromic systems and topological recursion, arXiv:1911.07739 [math-ph].

[81] H. Nagoya, Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations, *J. Math. Phys.* **56** (2015), 123505; arXiv:1505.02398.

[82] H. Nagoya, Remarks on irregular conformal blocks and Painlevé III and II tau functions; arXiv:1804.04782.

[83] P. Norbury, String and dilaton equations for counting lattice points in the moduli space of curves, *Trans. Amer. Math. Soc.*, **365** (2013), 1687–1709; arXiv:0905.4141 [math.AG].

[84] P. Painlevé, Sur les équations différentiel les du second ordre et d'ordre supérieur dont l'intégrale génŕale est uniforme. *Acta Mat.*, **25** (1902), 1–85.

[85] M. Sato, T. Aoki, T. Kwai and Y. Takei, Algebraic analysis of singular perturbations (特異摂動の代数解析), RIMS Kokyuroku, **750** (1991) 43–51 (notes by A. Kaneko in Japanese).

[86] D. Sauzin, Introduction to 1-summability and resurgence, arXiv:1405.0356 [math.DS]. (Part II in "Divergent Series, Summability and Resurgence I : Monodromy and Resurgence", by C. Mitschi and D. Sauzin, Springer, 2016.)

[87] K. Strebel, Quadratic Differentials, Springer–Verlag, 1984.

[88] Y. Takei Y, An explicit description of the connection formula for the first Painlevé equation, in *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, Kyoto University Press, Kyoto, 2000, 271–296.

[89] Y. Takei, Sato's conjecture for the Weber equation and transformation theory for Schrödinger equations with a merging pair of turning points, *RIMS Kôkyurôku Bessatsu*, **B10** (2008), pp. 205–224.

[90] Y. Takei, WKB analysis and Stokes geometry of differential equations, In "Analytic, Algebraic and Geometric Aspects of Differential Equations" (G. Filipuk, Y. Haraoka and S. Michalik eds.), Trends in Mathematics. Birkhäuser, Cham. 2017. (Preprint of RIMS-1848.)

[91] A. van Spaendonck and M. Vonk, Painlevé I and exact WKB: Stokes phenomenon for two-parameter transseries, *J. Phys. A: Math. Theor.*, **55** (2022), 454003; arXiv:2204.09062 [hep-th]

[92] A. Voros, The return of the quartic oscillator – The complex WKB method, *Ann. Inst. Henri Poincaré*, **39** (1983), pp. 211–338.

[93] J. Zhou, Intersection numbers on Deligne–Mumford moduli spaces and quantum Airy curve; arXiv:1206.5896.