Les Houches Lectures on
Exact WKB Method & Painlevé Equations

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Part I : Introduction to Exact WKB Method

- WKB solution, Spectral curve
- Stokes graph, Borel summability
- Voros connection formula and its application

Part II : Application to Painlevé Equations

- Review of Painlevé equations
- Topological recursion and τ-function

(with the viewpoint of topological recursion / quantum curve correspondence)

- Conjectural resurgent structure

Part I

Introduction to Exact WKB Method

Ref:

- T. Kawai & Y. Takei:
  Algebraic Analysis of Singular Perturbation Theory
  AMS Transl. 2005

- Y. Takei:
  WKB analysis and Stokes geometry of differential equations. 2017
  (RIMS-Preprint 1848)

Lecture 1

I-1: Schrödinger-type ODE and WKB solutions

$$\left(\hbar^2 \frac{d^2}{dx^2} - Q(x)\right) \psi(x,\hbar) = 0 \quad \cdots (\mathrm{Sch})$$

- $0 < \hbar \ll 1$ (perturbative parameter)
- $Q(x)$: rational function of x

Assumption

(i) $Q(x)$ has at least one zero.
All zeros are simple.

(ii) Meromorphic quadratic differential $\phi(x) = Q(x)\, dx^2$ (associated with (Sch)) has pole of order ≥ 2 at ∞.

- WKB (formal) solution

Take a new unknown function

$$\psi = \exp\left(\int_{x_0}^{x} P\, dx \right)$$

x_0 : generic pt

$$(\text{Sch}) \Rightarrow \hbar^2 \left(\frac{dP}{dx} + P^2 \right) = x \quad \cdots \quad (R)$$

Riccati eq

Put WKB-ansatz :

$$P(x, \hbar) = \sum_{m \geq -1} \hbar^m P_m(x)$$

$$\rightsquigarrow (R) \quad P_{-1}^2 = Q(x) \quad \rightsquigarrow \quad P_{-1}^{(\pm)}(x) = \pm \sqrt{Q(x)}$$

$$2 P_{-1} P_0 + \frac{dP_{-1}}{dx} = 0 \quad \rightsquigarrow \quad P_0^{(\pm)}(x) = - \frac{Q'(x)}{4Q(x)}$$

$$2 P_{-1} P_{m+1} + \sum_{\ell=0}^{m} P_\ell \cdot P_{m-\ell} + \frac{dP_m}{dx} = 0 \quad (m \geq 0)$$

" WKB recursion "

$\approx$ similar to topological recursion

$\rightsquigarrow$ Two formal solutions

term-wise integral

$$\begin{cases} \psi_\pm(x,\hbar) = \exp\left(\int_{x_0}^{x} P_\pm(x',\hbar)\, dx' \right) \\ P_\pm(x,\hbar) = \pm \hbar^{-1}\sqrt{Q(x)} - \dfrac{Q'(x)}{4Q(x)} + \cdots \end{cases}$$

normalization of WKB solution

$\hookrightarrow$ choice of path of integration

Example (& digression)

$$\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0 \quad : \text{Airy eq.}$$

$$Q(x) = x$$

$$\phi(x) = x\, dx^2 \underset{\substack{\varphi \\ x = 1/w}}{=} \frac{1}{w}\left(-\frac{dw}{w^2}\right)^2 = \frac{dw^5}{w^5}$$

: $w=0$ ($x=\infty$) is a pole of order 5

WKB-rec

$$\rightsquigarrow \quad p_{-1}^{(\pm)} = \pm\sqrt{x}, \quad p_0^{(\pm)} = -\frac{1}{4x},$$

$$p_1^{(\pm)} = \mp\frac{5}{32}x^{-\frac{5}{2}}, \quad p_2^{(\pm)} = -\frac{15}{64}x^{-4}, \cdots$$

$$\rightsquigarrow \quad \psi_{\pm}^{\text{Airy}}(x,\hbar)$$

normalized at $x_0 = \infty$

$$= \exp\left(\pm\frac{2}{3}x^{\frac{3}{2}}\cdot\hbar^{-1} - \frac{1}{4}\log x \pm \underbrace{\frac{5}{48}x^{-\frac{3}{2}}\hbar + \cdots}\right)$$

__Fact__ [J, Zhou 12] etc.

$$\psi_{\pm}^{\text{Airy}} = \exp\left(\sum_{\substack{g\geq 0\\ n\geq 1}} \frac{(\pm\hbar)^{2g-2+n}}{n!}\int_\infty^{z(x)}\cdots\int_\infty^{z(x)} \omega_{g,n}^{\text{Airy}}(z_1,\cdots,z_n)\right)$$

↑ need regularization at $(g,n) = (0,1), (0,2)$

where $\omega_{g,n}^{\text{Airy}}(z_1,\cdots,z_n)$ is topological recursion correlator of Airy spectral curve

$$y^2 - x = 0 \quad \longleftrightarrow \quad (C, x, y) = (\mathbb{P}^1, z^2, z)$$

WKB recursion $\longleftrightarrow$ TR "quantum curve"

$P_m(x)$ are defined on double covering Riemann surface of x-plane. The geometry of the RS is important.

Def

- $\Sigma := \{ (x,y) \in \mathbb{C}^2 ; \ y^2 = Q(x) \}$ is called (WKB) spectral curve.

$$\begin{array}{ccccc} (x,y) \in \Sigma & \hookrightarrow & \overline{\Sigma} & \text{compactification} \\ \big\downarrow & \pi \big\downarrow 2:1 & \pi \big\downarrow 2:1 & \\ x \in \mathbb{C} & \hookrightarrow & \mathbb{P}^1 & \end{array}$$

- A turning point of (Sch) is either a zero or a simple pole of $\phi(x) = Q(x)dx^2$.

[Koike 00], [Hollands-Neitzke 16]

P_m's are meromorphic function on $\overline{\Sigma}$.

In particular, WKB solutions are not defined at turning points.

Also, WKB solutions are usually divergent.

Prop

$\forall K \underset{\text{cpt}}{\subset} \mathbb{C} \setminus \{\text{poles \& zeros } \phi\}$

$\exists C_K, r_K > 0$ s.t.

$$\sup_{x \in K} \left| P^{(\pm)}_{\pm, m}(x) \right| \leq C_K \cdot r_K^m \cdot m! \qquad (m \geq 0)$$

Gevrey 1 series

→ Let's take Borel sum w.r.t $\hbar$!

I-2: Borel summability (w.r.t $\hbar$)

Quick review

formal series // $\int_0^{+\infty} e^{-\zeta/\hbar} \frac{\zeta^m}{m!} d\zeta$ (Recall: $0 < \hbar \ll 1$)

$$f(\hbar) = \sum_{m \geq 0} \hbar^{m+1} f_m \, . \quad |f_m| \sim m!$$

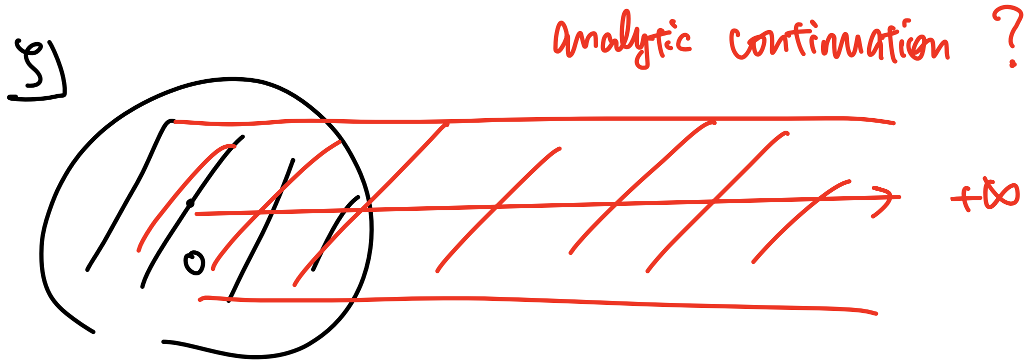
term-wise $\mathcal{L}^{-1}$

$$\longmapsto \quad (\mathcal{B}f)(\zeta) = f_B(\zeta) := \sum_{m \geq 0} \frac{\zeta^m}{m!} f_m \ : \ \text{Borel transform of } f$$

$$\xrightarrow{\text{A.C } \& \ \mathcal{L}} \quad (\mathcal{S}f)(\hbar) = \int_0^{+\infty} e^{-\zeta/\hbar} f_B(\zeta) \, d\zeta \ : \ \text{Borel sum of } f$$

$$\underset{\hbar \to +0}{\sim} \ f(\hbar) \quad (\text{Watson's lemma})$$

Main issue :



In the exact WKB (in 2nd order),
the following geometric object controls the summability.

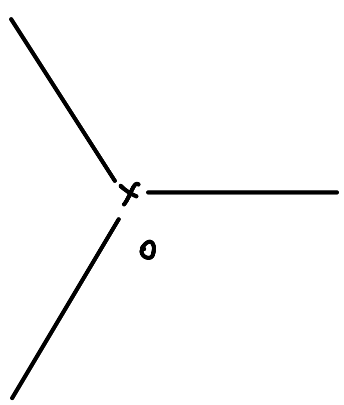
Def Spectral network associated with $\phi = Q(x)dx^2$
$\parallel$
The Stokes graph of (Sch) is a graph on x-plane with

- vertices ... zeros and poles of ϕ
- edges ... Stokes curves (= horizontal trajectories) emanating from a turning point v and defined by $\mathrm{Im} \int_v^x \sqrt{Q(x)}\,dx = 0$

If $\arg \hbar = \theta$, then we put $e^{-i\theta}$ there

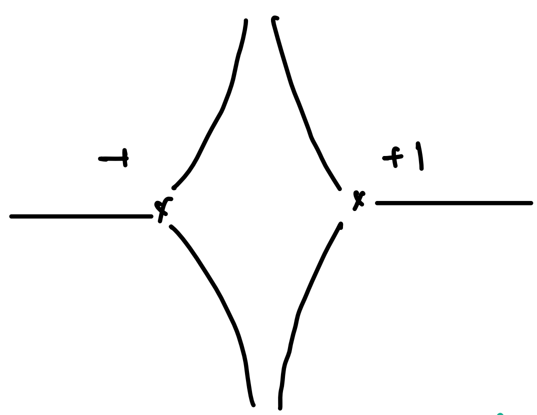
Examples

$Q(x) = x$: Airy



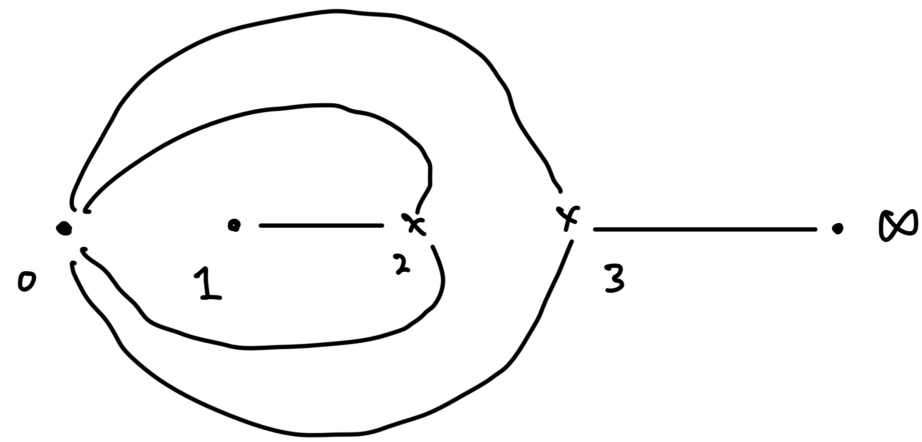
order 5 at ∞

$Q(x) = x^2 - 1$: Weber

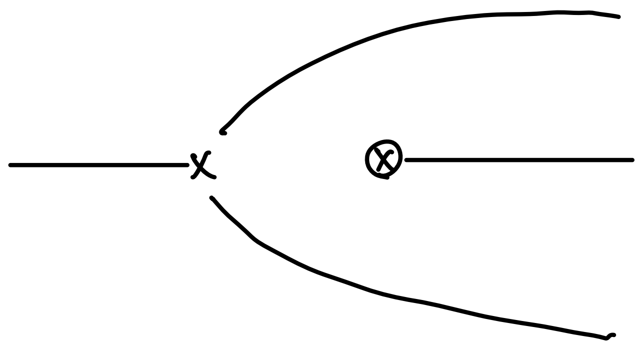


order 6 at ∞

$$Q(x) = \frac{(x-2)(x-3)}{x^2(x-1)^2} \quad : \text{hypergeometric}$$ $\leftarrow$ ϕ has order 2 pole at $x=\infty$



$$Q(x) = \frac{x+1}{x}$$ order 4 at ∞



$\times$: simple zero

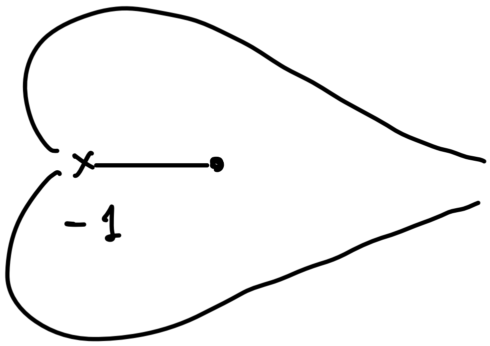
$\otimes$: simple pole

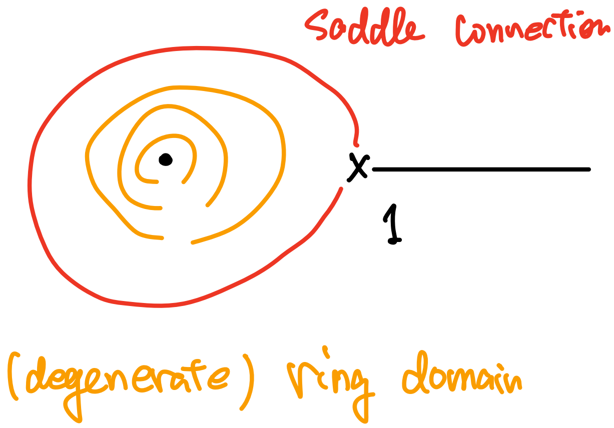
$\bullet$: pole of order ≥ 2

In general :

- 3 (resp 1) Stokes curves emanate from $\times$ (resp $\otimes$).
- Near a pole of ϕ with order m (≥ 2) :
  - If $m \geq 3$, then $\exists\,(m-2)$ asymptotic directions
    ($\leftrightarrow$ singular directions around irregular sing. pt)
  - If $m = 2$, it depends on phase of $\operatorname{Res}_{x=p} \sqrt{\phi} \in \mathbb{C}$

e.g., $Q(x) = \frac{x+1}{x^2}$ $\qquad$ $Q(x) = \frac{x-1}{x^2}$

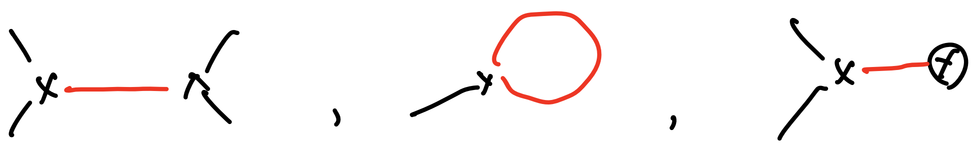




[Dunster–Lutz–Schäfke 93], [Koike–Schäfke]
[Nemes 20], [Nikolaev 21], ...

Thm

If the Stokes graph doesn't contain saddle conn.

(e.g.,  etc)

then the WKB solutions are Borel summable on each Stokes region (i.e., face of Stokes graph)

$\Psi_\pm := \mathcal{S}\psi_\pm$ are basis of hol solutions of (Sch) defined on the Stokes region

Idea of proof

[Koike – Schäfke], [Takei 17]

$$P(x;\hbar) = \hbar^{-1} \underbrace{P_{-1}(x)}_{\sqrt{Q(x)}} + P_0(x) + \underbrace{\hbar P_1(x) + \cdots}_{=: \hbar^{-1} T}$$

$\uparrow$

$$\hbar^2 \left(P^2 + \frac{dP}{dx} \right) = Q(x)$$

$$\left(\text{i.e., } \begin{array}{l} T(x,\hbar) = \hbar^2 P_1(x) + \cdots \\ T_B(x,\zeta) = P_1(x)\zeta + \cdots \end{array} \right)$$

$$P^2 = \cancel{\hbar^{-2} Q} + P_0^2 + \hbar^{-2} T^2$$

$$+ 2\hbar^{-2}\sqrt{Q}\cdot T + \cancel{2\hbar^{-1}\sqrt{Q}\cdot P_0} + 2\hbar^{-1} P_0 \cdot T$$

$$\frac{dP}{dx} = \cancel{\hbar^{-1}\frac{dP_{-1}}{dx}} + \frac{dP_0}{dx} + \hbar^{-1}\cdot\frac{dT}{dx}$$

$$\frac{Q(x)}{\hbar^2} = \cancel{\hbar^{-2} Q(x)}$$

$\Big\} \times \hbar^2$

$$\frac{2\sqrt{Q}}{\hbar}\cdot T + \frac{dT}{dx} = -\hbar^{-1}\cdot T^2 - 2P_0 T - \hbar\left(P_0^2 + \frac{dP_0}{dx}\right)$$

$$\Big\downarrow \quad \mathcal{B}: \quad \hbar^{m+1} \mapsto \frac{\xi^m}{m!}, \qquad \times\frac{1}{\hbar} \mapsto \frac{\partial}{\partial \xi}$$

$$2\sqrt{Q}\cdot\frac{\partial T_B}{\partial \xi} + \frac{\partial T_B}{\partial x}$$

$$\int_0^{\xi} T_B(\eta)\cdot T_B(\xi-\eta)\,d\eta$$

convolution product

$$= -\frac{\partial}{\partial \xi}\, T_B * T_B(\xi) - 2P_0\cdot T_B(\xi) - \left(P_0^2 + \frac{dP_0}{dx}\right)$$

$$\Big\downarrow \quad z = \int^x \sqrt{Q}\,dx \quad : \text{ Liouville transform}$$

$$2\frac{\partial T_B}{\partial \xi} + \frac{\partial T_B}{\partial z} = A_1(z)\cdot\frac{\partial}{\partial \xi} T_B^{*2} + A_2(z)\, T_B + A_3(z)$$

where $A_1 = -\dfrac{1}{\sqrt{Q}}$, $\quad A_2 = -\dfrac{2P_0}{\sqrt{Q}}$

$$A_3 = -\frac{1}{\sqrt{Q}}\left(P_0^2 + \frac{dP_0}{dx}\right)$$

method of characteristics $(+\infty)$

$$\boxed{\begin{aligned} T_B(z,\zeta) = &\int_0^{\zeta} A_1\left(z-\frac{\zeta-t}{2}\right) \frac{\partial T_B^{*2}}{\partial \zeta}\left(z-\frac{\zeta-t}{2}, t\right) dt \\ &+ \int_0^{\zeta} A_2\left(z-\frac{\zeta-t}{2}\right) \cdot T_B\left(z-\frac{\zeta-t}{2}, t\right) dt \\ &+ \int_0^{\zeta} A_3\left(z-\frac{\zeta-t}{2}\right) dt \end{aligned}}$$

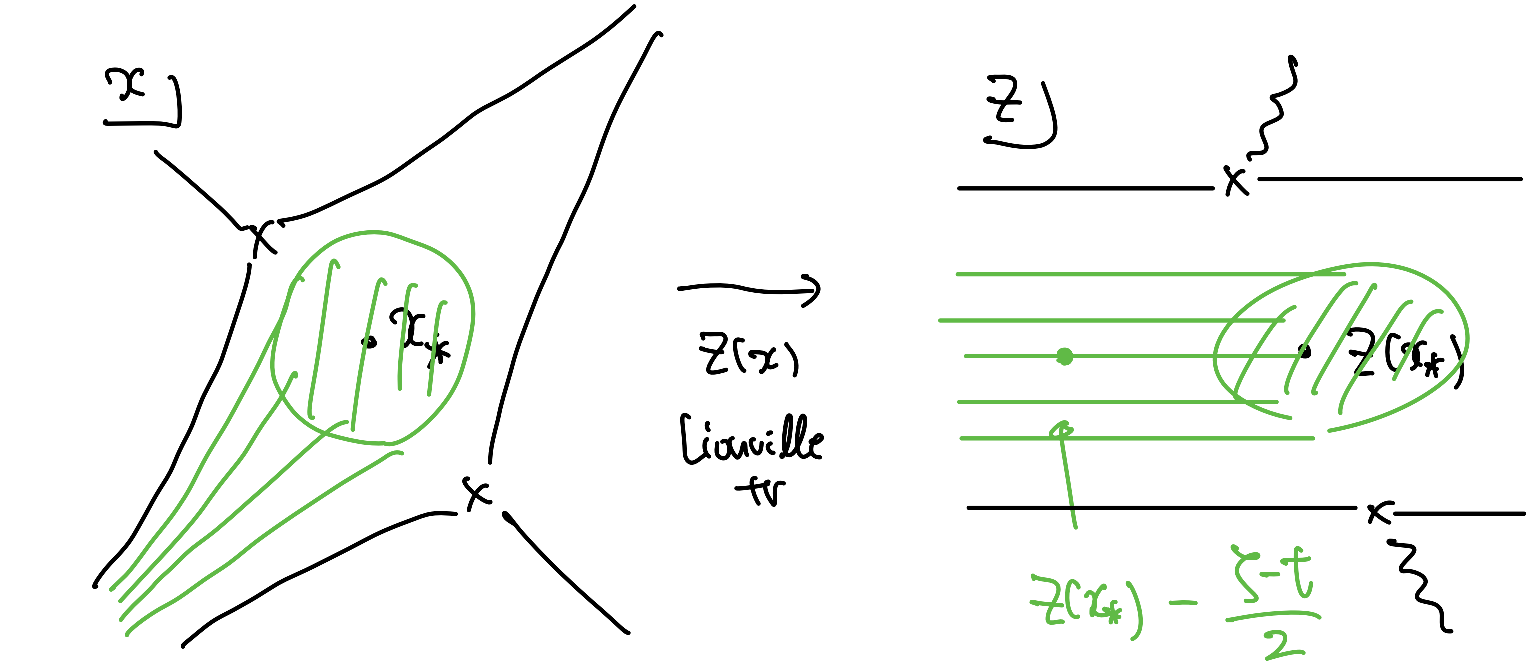
T_B is unique hol. sol. (at $\zeta=0$)

satisfying $T_B(z,0)=0$, $\partial_\zeta T_B(z,0)=P_1$

To take $\mathrm{Re}\,\zeta \to +\infty$,

we need information (growth of $A_i(z)$)

when $\mathrm{Re}\, Z = \mathrm{Re} \int^x \sqrt{Q}\, dx \to -\infty$



when $\mathrm{Re}\,\zeta \gg 1$

family of negative trajectories

- x_* & Stokes graph $\rightsquigarrow$ 

P : pole of order ≥ 2 where $A_i(z)$ are "bounded"

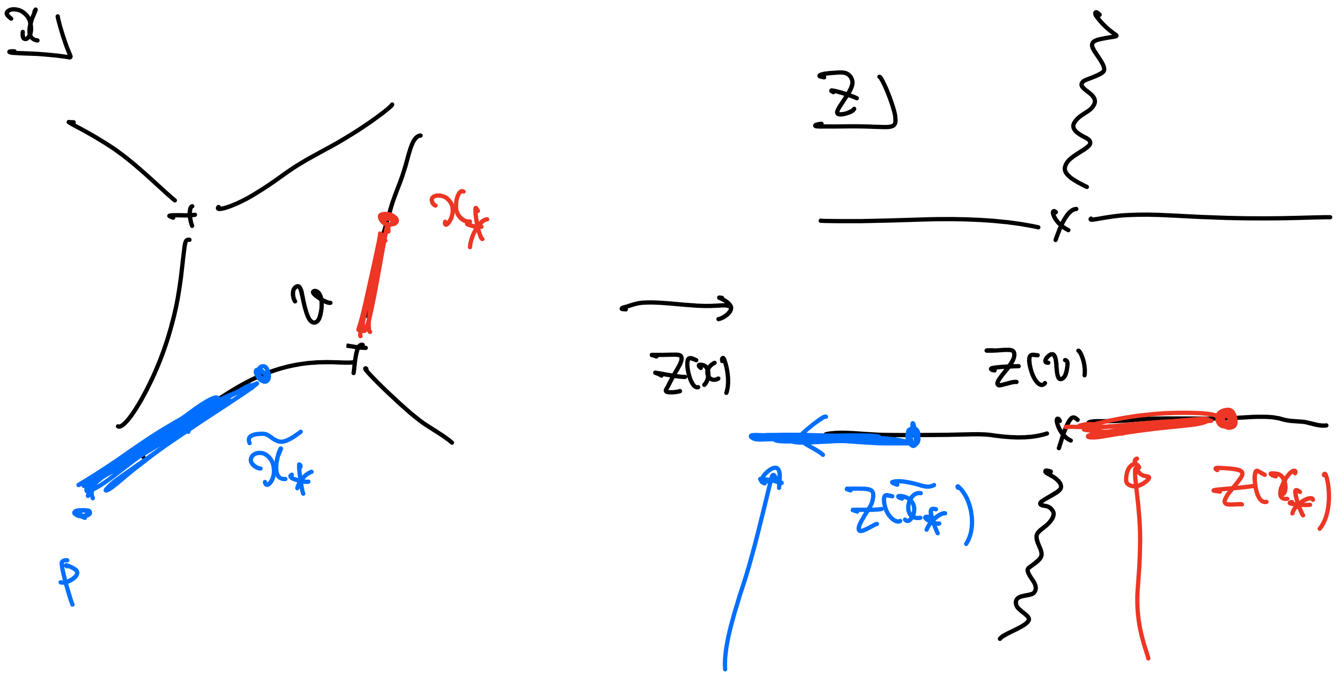
Need careful discussion if pole order $= 2$ $\rightarrow$

$\rightarrow$ Iteration (+ estimation) works!

$\rightarrow$ P is Borel summable!

- $x_* \in$ positive / negative part of a Stokes curve :

$\rightsquigarrow$ Iteration does not / does work ... !



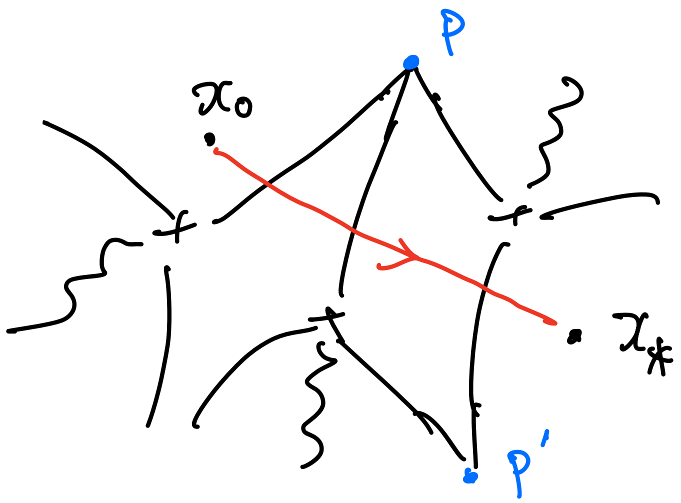
$z(\tilde{x}_*) - \frac{\zeta - t}{2}$ never meets with singular pts of Ai

$\rightsquigarrow$ P : summable

$z(x_*) - \frac{\zeta - t}{2}$ will hit a singular point of Ai ($\leftrightarrow$ turning point)

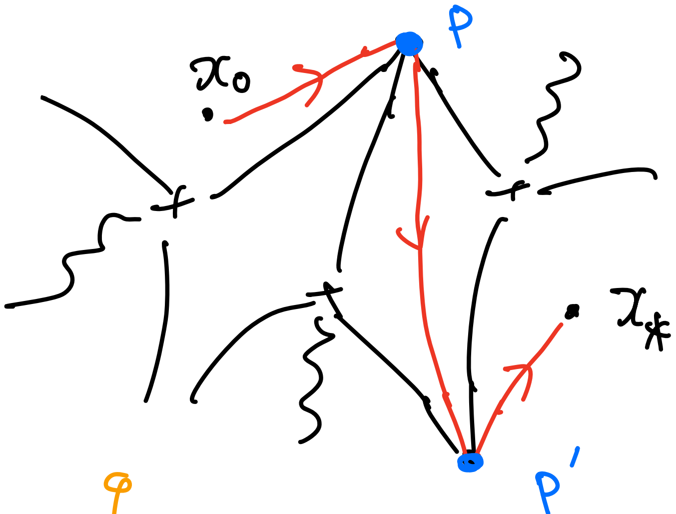
$\rightsquigarrow$ We can NOT prove the summability of P.

For WKB solution $\psi_\pm = \exp\left(\int_{x_0}^{x} P_\pm(x',\hbar)\, dx' \right)$



p, p': singular point of (Sch)

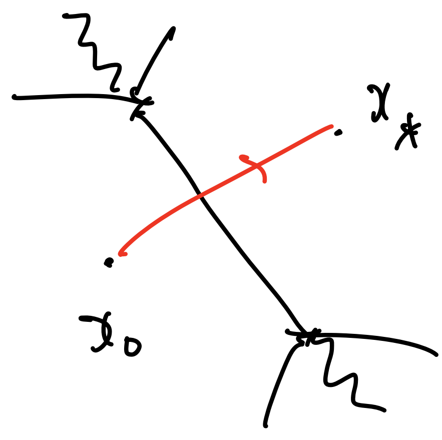
|| as formal $\hbar$-series "Koike's trick"



We can take p, p' as endpoints for integral of P_m ($m \geq 1$)

$P_\pm$ are Borel summable at any point on the deformed path!

But this deformation is NOT allowed if $\exists$ saddle conn.

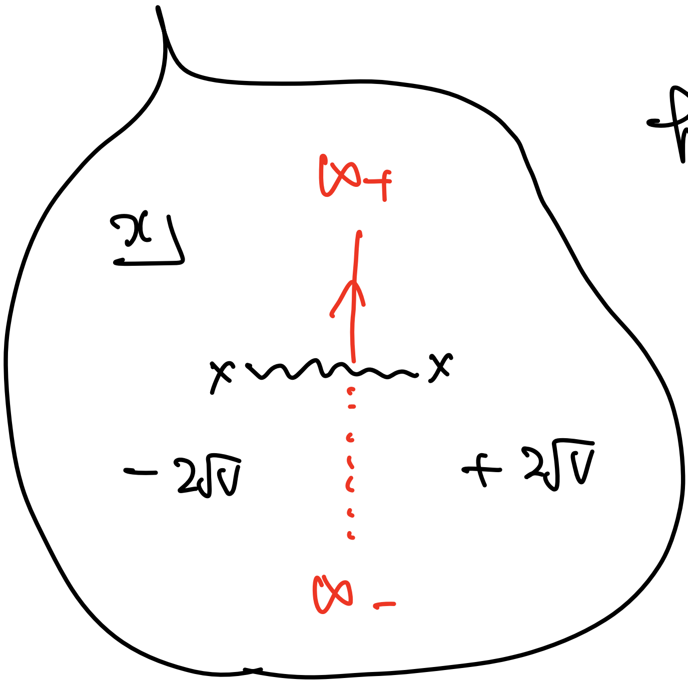


Excercise

$$\left(\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \nu\right)\right)\psi = 0 \; : \; \text{Weber eq. } (\nu \in \mathbb{C}^*)$$

Prove (or check by computer) that

$$\int_{\infty_-}^{\infty_+} P_m^{\mathrm{Web}}(x)\,dx = \begin{cases} 0 & (m = 2k : \text{even}) \\ \dfrac{(1-2^{1-2k})\cdot B_{2k}}{2k(2k-1)\cdot \nu^{2k-1}} & (m = 2k-1 : \text{odd}) \end{cases}$$



hold for $m \geq 1$ (i.e., $k \geq 1$)

$$V_\infty = \int_{\infty_-}^{\infty_+} P(x,\hbar)\,dx$$

: "Voros period" of Weber eq

Borel sing $\longleftrightarrow$ $2\pi i \nu = \oint_\gamma y\,dx$

(c.f., Andy's comment)

$$\mathrm{TR} : F_g^{\mathrm{Web}} = \frac{\chi(M_g)}{\nu^{2g-2}}$$

Lecture 2

I-3: Connection formula

Assumption

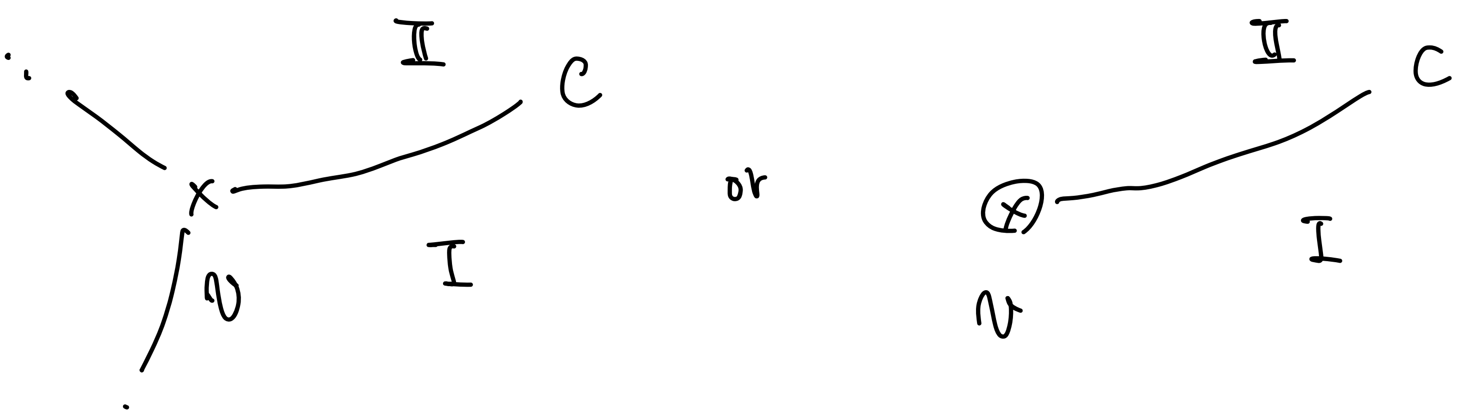
- Stokes graph of (Sch) doesn't contain any saddle connection
- v : a turning point

i.e., simple zero or simple pole of Q

Remark

In this case, we allow

$$Q = \underbrace{Q_0(x)}_{\text{has simple pole at } v} + \hbar^2 \underbrace{Q_2(x)}_{\text{has at most double pole at } v}$$



- I, II : adjacent Stokes regions,

  have a Stokes curve C as common boundary.

  $\left(\mathrm{II} \xleftarrow[v]{} \mathrm{I} \ : \text{counter-clockwise} \right)$

- We normalize the WKB solution at v:

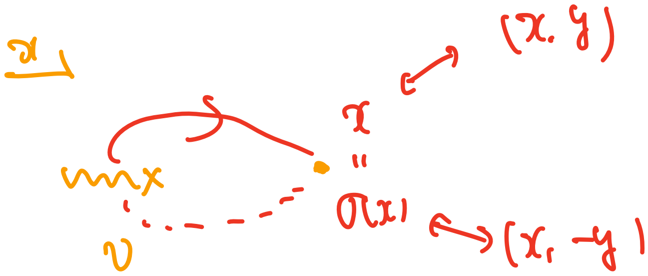
$$\psi_\pm(x,\hbar) \ \text{``=''} \ \exp\left(\int_v^x P^{(\pm)}(x',\hbar)\, dx' \right)$$

More precisely:

Take odd/even decomposition and define

$$\psi_\pm = \frac{1}{\sqrt{P_{odd}}} \exp\left(\pm \int_v^x P_{odd}(x;\hbar)\, dx \right)$$

where $\displaystyle \int_v^x P_{odd}\, dx := \frac{1}{2} \int_{\sigma(x)}^x P_{odd}\, dx$



- $\Psi_\pm^J$: the Borel sum of $\Psi_\pm$ on region J.

connection matrix

Since these are hol. sol. of (Sch), there exists invertible 2×2 matrix S satisfying

$$\left(\Psi_+^{I}, \Psi_-^{I}\right) = \left(\Psi_+^{II}, \Psi_-^{II}\right) \cdot S$$

analytic continuation across C

where

$$S = \begin{cases} \begin{pmatrix} 1 & 0 \\ i\mu & 1 \end{pmatrix} & \text{if } \int_v^x \sqrt{Q}\, dx > 0 \quad \text{on } C \\ \\ \begin{pmatrix} 1 & i\mu \\ 0 & 1 \end{pmatrix} & \text{if } \int_v^x \sqrt{Q}\, dx < 0 \quad \text{on } C \end{cases}$$

for some $\mu \in \mathbb{C}$. Stokes multiplier (2d BPS index)

Thm

(i) If v is a simple zero, then $\mu = 1$

[Voros 83, Aoki-Kawai-Takei 91, ..., Kamimoto-Koike 12]

(ii) If v is a simple pole, then

$$\mu = 2\cos\left(\pi\sqrt{1+4A}\right)$$

where $A = \lim_{x \to v} (x-v)^2 Q_2(x)$

[Koike 00]

Remarks

- This can be understood as Stokes phenomenon w.r.t $\hbar$.

  $\Psi_+^{\mathrm{I}}, \Psi_+^{\mathrm{II}} \sim \psi_+ \quad \hbar \to +0$ (Watson's lemma)

  and $\Psi_+^{\mathrm{I}} - \Psi_+^{\mathrm{II}} = i\mu\, \Psi_-^{\mathrm{II}}$: exp. small

- (i) $\Leftrightarrow$ path-lifting of [Gaiotto-Moore-Neitzke 12]
  (We will make it more precise below)

- The same connection formula is valid when we fix x and vary $\vartheta = \arg\hbar$.

<u>Idea of proof</u> (v = simple zero)

- Airy case $\longrightarrow$ follows from properties of ${}_2F_1$ (c.f., Aniceto's lecture)

- General case : Use Exact WKB-theoretic transformation

$\exists\, X(x,\hbar) = \sum_{m\geq 0} \hbar^m X_m(x)$: formal change of coordinate

s.t.,

$$\psi_\pm(x,\hbar) = \left(\frac{\partial X}{\partial x}(x,\hbar)\right)^{-\frac{1}{2}} \psi_\pm^{\mathrm{Airy}}(X(x,\hbar),\hbar)$$

normalized at v

$$\left(\hbar^2 \frac{d^2}{dX^2} - X\right)\psi_\pm^{\mathrm{Airy}} = 0$$

normalized at $X = 0$

$$\text{R.H.S} = \left(\frac{\partial X}{\partial x}\right)^{-\frac{1}{2}} \cdot \sum_{k=0}^{\infty} \frac{\Xi^{k}}{k!} \cdot \left[\frac{\partial^{k}}{\partial X^{k}} \psi_{I}^{\text{Airy}}(X;\hbar) \right]_{X = X_0(x)}$$

$\Big\downarrow$ Borel tr. $\quad \left(\text{where } \Xi = X - X_0 = \sum_{m \geq 1} \hbar^{m} X_m(x) \right)$

$$\psi_{I,B}(x,\xi) = \int_{\mp A(x)}^{\xi} K\left(x, \xi - \xi', \frac{\partial}{\partial x}\right) \cdot \psi_{I,B}^{\text{Airy}}(X_0(x), \xi')\, d\xi'$$

This has no singularity on the positive real axis on ξ-plane

$\rightsquigarrow$ preserve the singularity structure on ξ-plane

"micro-differential operator"

Thus we can compute alien derivative of $\psi_{I,B}$

from that of $\psi_{I,B}^{\text{Airy}}$!

[Aoki-Kawai-Takei 91, Kamimoto-Koike 12] //

Remark

quantization of $QH^*(\mathbb{CP}^1)$ $\quad p^2 - e^t = 0$

$$\begin{array}{c} p \\ \Downarrow \\ \psi \end{array} \qquad \begin{array}{c} e^t \\ \Downarrow \\ x \end{array}$$

Quantum differential equation for $\mathbb{CP}^1$:

$$\left[\left(\hbar x \frac{d}{dx}\right)^2 - x\right]\Psi = 0$$

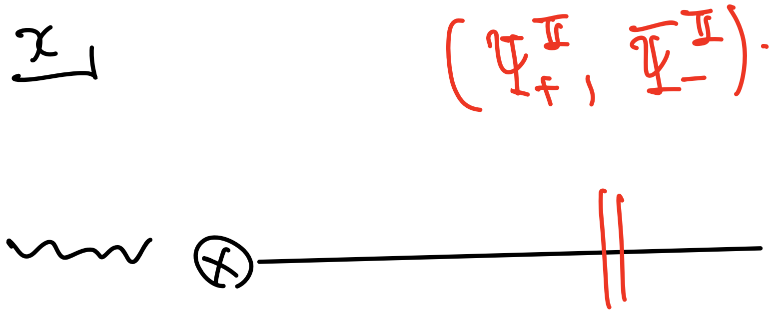
& satisfied by $\int e^{\frac{1}{\hbar}W(\phi,x)}\frac{d\phi}{\phi}$

with $W = \phi + \dfrac{x}{\phi}$

$$\Longleftrightarrow \quad \left[\hbar^2 \frac{d^2}{dx^2} - \left(\frac{1}{x} - \frac{\hbar^2}{4x^2}\right)\right]\tilde{\Psi} = 0$$

gauge transf.

Koike's connection formula gives



$$(\Psi_+^{II}, \Psi_-^{II}) \cdot \begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix} = (\Psi_+^{I}, \Psi_-^{I})$$

$$2 = \chi(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))$$

Consistent with the Dubrovin's conjecture !

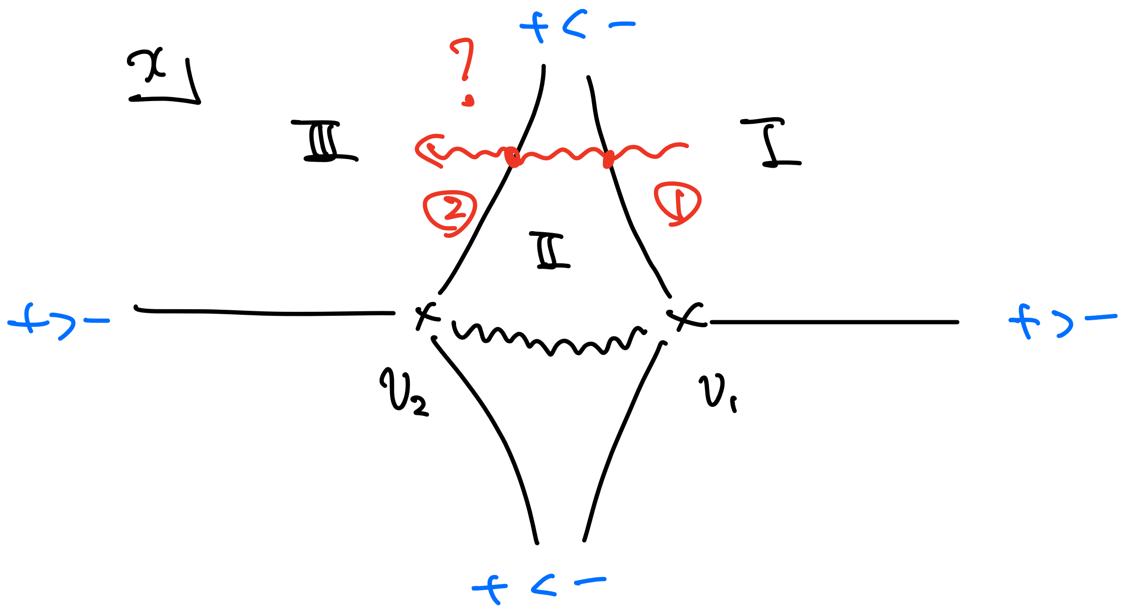
[Dubrovin 1989], [Guzzetti 1999], ...

I-4 : Application to monodromy computation

Example : Weber equation

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \nu\right)\right] \psi(x,\hbar) = 0$$

Suppose $\nu \in \mathbb{R}_{>0}$. The Stokes graph is



$$\psi_{\pm} = \psi_{\pm}^{v_1} = \frac{1}{\sqrt{P_{odd}}} \exp\left(\pm \int_{v_1}^{x} P_{odd}\, dx\right)$$: normalized at v_1

Let us compute connection formula from I → III.

- Connection formula at ① :

Previous Voros' formula is applicable

$$\left(\psi_{+}^{\mathrm{I}}, \psi_{-}^{\mathrm{I}}\right) = \left(\psi_{+}^{\mathrm{I}}, \psi_{-}^{\mathrm{I}}\right) \cdot \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

- Connection formula at ② :

Voros formula doesn't hold for $\psi_{\pm}$ at ②

since the normalization point is different from v_2.

However :

$$\psi_{\pm}^{v_1} = \frac{1}{\sqrt{P_{odd}}} \exp\left(\pm \underbrace{\int_{v_1}^{x}}_{\int_{v_1}^{v_2} + \int_{v_2}^{x}} P_{odd}\, dx \right)$$

$$= \exp\left(\pm \underbrace{\int_{v_1}^{v_2} P_{odd}\, dx}_{\frac{1}{2}\oint_{\gamma} P_{odd}\, dx} \right) \cdot \underbrace{\psi_{\pm}^{v_2}}_{\text{Voros formula is applicable at ②}}$$

$\gamma \in H_1(\Sigma;\mathbb{Z})$

$$\therefore \quad \psi_\pm^{v_1} = e^{\pm\frac{1}{2}V_\gamma}\,\psi_\pm^{v_2} \quad \text{where} \quad V_\gamma = \oint_\gamma P_{odd}\,dx \,.$$

This is also Borel summable under saddle-free assumption

(Koike's trick)

$$\therefore \quad (\psi_+^{I}, \psi_-^{I})$$
$$= (\psi_+^{III}, \psi_-^{III}) \cdot \begin{pmatrix} e^{\frac{1}{2}V_\gamma} & 0 \\ 0 & e^{-\frac{1}{2}V_\gamma} \end{pmatrix}^{-1} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}V_\gamma} & 0 \\ 0 & e^{-\frac{1}{2}V_\gamma} \end{pmatrix}$$
$$= (\psi_+^{III}, \psi_-^{III}) \cdot \begin{pmatrix} 1 & i\,e^{-V_\gamma} \\ 0 & 1 \end{pmatrix}$$

understood as Borel sum

① & ② implies

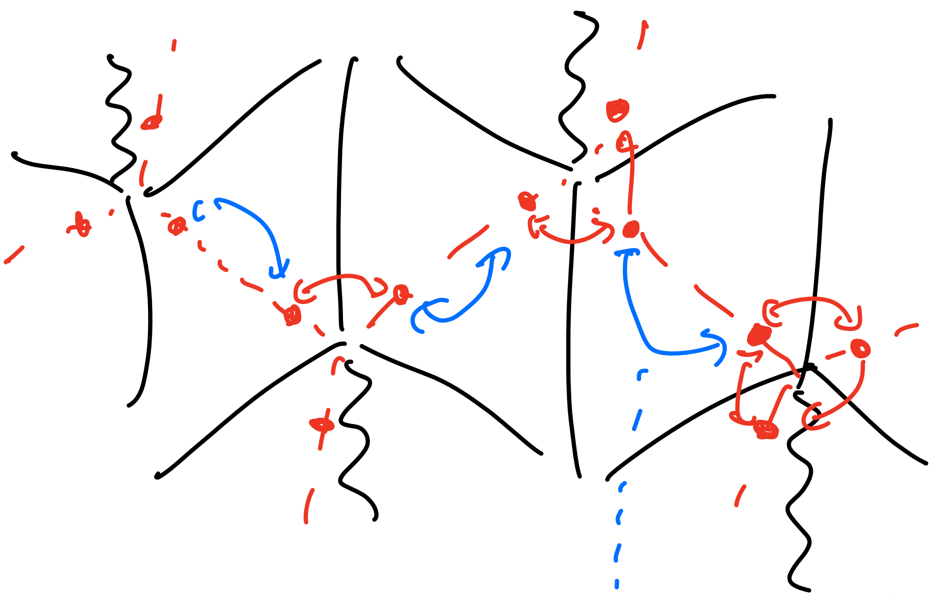
$$(\psi_+^{I}, \psi_-^{I}) = (\psi_+^{III}, \psi_-^{III}) \cdot \begin{pmatrix} 1 & i\,(1 + e^{-V_\gamma}) \\ 0 & 1 \end{pmatrix}$$

Thus we have observed that
A.C. of Borel resummed WKB solutions
are described by <u>periods</u> of P_{odd}.

<u>Thm</u> (Sato - Aoki - Kawai - Takei 91 : RIMS Kokyuroku 750)

If there is no saddle connection the Stokes graph,
then the monodromy / Stokes / Connection matrices
are explicitly described by the Borel sum of

"Voros symbols" $\exp\Big(\underbrace{\oint_\gamma P_{\text{odd}}\,dx}_{\text{"Voros period"}}\Big)$ $\quad (\gamma \in H_1(\Sigma;\mathbb{Z}))$

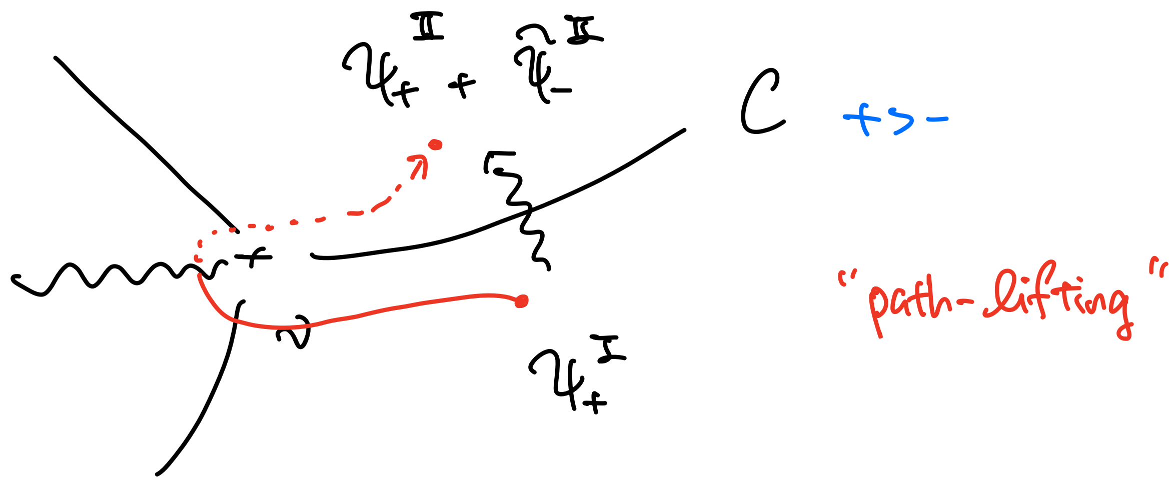


Airy conn. matrix $\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}^{\pm 1}$ or $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}^{\pm 1}$

Voros symbols $\begin{pmatrix} e^{\frac{1}{2}V_\gamma} & 0 \\ 0 & e^{-\frac{1}{2}V_\gamma} \end{pmatrix}$

Remark

The Voros connection formula can be alternatively formulated as follows:



where $\tilde{\psi}_-^{\mathrm{II}}$ is the Borel sum of $\tilde{\psi}_-$ which is obtained by term-wise analytic continuation of ψ_+ along "detoured path" shown above.

$$\because \quad \exp\left(\pm \int_v^x P_{\mathrm{odd}}\, dx\right) \mapsto \exp\left(\mp \int_v^x P_{\mathrm{odd}}\, dx\right)$$

$$\frac{1}{\sqrt{P_{\mathrm{odd}}}} = \frac{\hbar^{\frac{1}{2}}}{\sqrt{P_{-1}}} \underbrace{\left(1 + \hbar\left(-\frac{1}{2}\frac{P_1}{P_{-1}}\right) + \cdots\right)}_{\text{single valued at } v}$$

$$\left(\frac{\hbar^{1/2}}{(x-v)^{1/4}} \doteq\right) \frac{\hbar^{\frac{1}{2}}}{\sqrt[4]{Q_0}} \mapsto \underbrace{e^{-2\pi i \cdot \left(-\frac{1}{4}\right)}}_{= +i} \frac{\hbar^{\frac{1}{2}}}{\sqrt[4]{Q_0}} \quad //$$

Remark (2d/4d wall-crossing formula [Gaiotto–Moore–Neitzke 2011])

Example : $Q(x) = \frac{x^2}{4} - 1$ (Weber eq with $\nu = 1$)

Fix x and compute "total Stokes matrix"

$$S^{x}_{tot} = \prod^{\curvearrowleft}_{0 \le \theta < \pi} S^{x}_{\theta}$$

either Voros connection matrix or DDP matrix

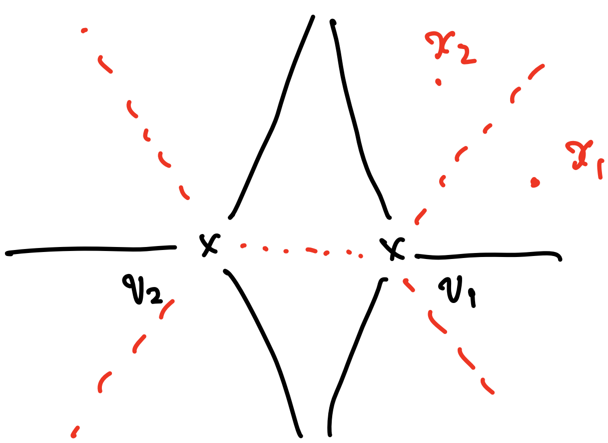
$$\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}^{-1} \text{ or } \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}^{-1} \qquad \begin{pmatrix} (1+e^{V_\gamma})^{1/2} & 0 \\ 0 & (1+e^{V_\gamma})^{-1/2} \end{pmatrix}^{-1}$$

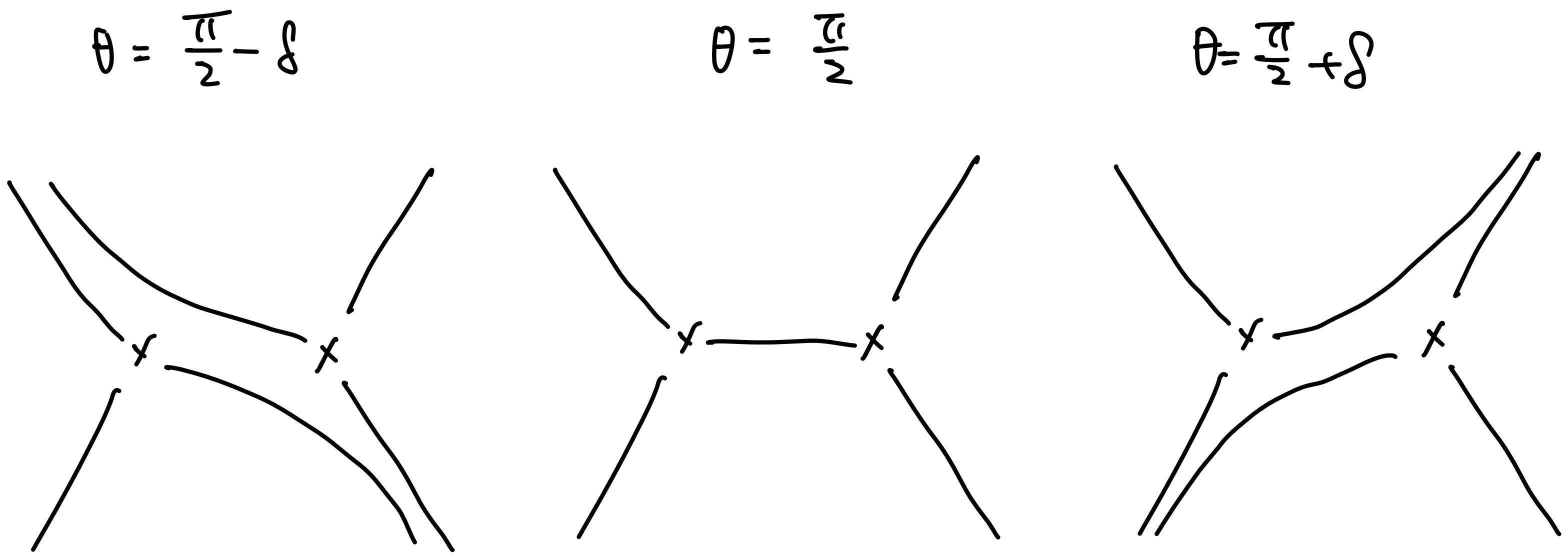
NOTE: We use connection formulas in opposite direction.

In other words,

$$\left(\Psi_+^{(\theta=0)} , \Psi_-^{(\theta=0)} \right) = \left(\Psi_+^{(\theta=\pi)} , \Psi_-^{(\theta=\pi)} \right) \cdot S^{x}_{tot}$$

$\theta = 0$ (and π)





We normalize the WKB solution at v_1. Then,

$$S^{x_1}_{tot} = \overbrace{\begin{pmatrix} (1+e^{V_\gamma})^{\frac{1}{2}} & 0 \\ 0 & (1+e^{V_\gamma})^{-\frac{1}{2}} \end{pmatrix}^{-1}}^{4d} \cdot \overbrace{\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}^{-1}}^{2d}$$

$$\parallel$$

$$\begin{pmatrix} (1+e^{V_\gamma})^{-\frac{1}{2}} & 0 \\ -i(1+e^{V_\gamma})^{\frac{1}{2}} & (1+e^{V_\gamma})^{\frac{1}{2}} \end{pmatrix}$$

$$\parallel$$

$$S^{x_2}_{tot} = \underbrace{\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}^{-1}}_{2d} \underbrace{\begin{pmatrix} 1 & 0 \\ ie^{V_\gamma} & 1 \end{pmatrix}^{-1}}_{2d} \underbrace{\begin{pmatrix} (1+e^{V_\gamma})^{\frac{1}{2}} & 0 \\ 0 & (1+e^{V_\gamma})^{-\frac{1}{2}} \end{pmatrix}^{-1}}_{4d}$$

S^{x}_{tot} is locally constant under the variation of x.

$\rightsquigarrow$ 2d/4d wall-crossing formula (non-trivial relation among μ_{2d}, Ω_{4d})

Part II

Application to Painlevé Equations

Ref:

- A. Fokas – A. Its – A. Kapaev – V. Novokshenov : Painlevé Transcendents, AMS 2006.
- K. I : CMP 2019 (1902.06439)
- K. I – M. Mariño : SIGMA 2024 (2307.02080)

Lecture 3

II-1 : Motivation

Painlevé equations ← Painlevé, Gambier around 1900.

- Painlevé property : movable sing must be a pole.

e.g., $\frac{dq}{dt} = q^2 \rightsquigarrow q(t) = \frac{1}{c-t}$

has singularity at $t=c$ "movable sing"

✗ $\frac{d^2q}{dt^2} = \left(\frac{dq}{dt}\right)^2 \rightsquigarrow q(t) = -\log(c-t) + c'$

movable branch pt

$$P_{\mathrm{I}} : \frac{d^2q}{dt^2} = 6q^2 + t$$

$$P_{\mathrm{II}} : \frac{d^2q}{dt^2} = 2q^3 + tq + \theta$$

$$P_{\mathrm{III}} : \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\cdot\frac{dq}{dt} + \frac{q^3}{t^2} - \frac{\theta_\infty q^2}{t^2} + \frac{\theta_0}{t} - \frac{1}{q}$$

$\vdots$

These non-linear ODE has many nice properties

(i) τ-function (analogue of ϑ-function)

(ii) isomonodromy deformation (integrability) etc.

(i) $\dfrac{d^2q}{dt^2} = 6q^2 + t$ $\quad\longleftarrow\cdots\cdots$ analogue (non-autonomous version) $\cdots\cdots\quad$ elliptic function $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$

$\left(\wp'' = 6\wp^2 - \frac{1}{2}g_2 \right)$

$q = -\dfrac{d^2}{dt^2}\log\tau \quad \updownarrow$

$\updownarrow \quad \wp = -\dfrac{d^2}{dz^2}\log\sigma$

τ-function (entire) $\quad\longleftarrow\cdots\cdots$ analogue $\cdots\cdots\quad$ σ-function (ϑ-function) (entire)

(ii)
$$\begin{cases} \hbar \dfrac{\partial \Phi}{\partial x} = A\Phi, & A = \begin{pmatrix} p & 4(x-q) \\ x^2 + qx + q^2 + \frac{t}{2} & -p \end{pmatrix} \\[2ex] \hbar \dfrac{\partial \Phi}{\partial t} = B\Phi, & B = \begin{pmatrix} 0 & 2 \\ x + \frac{q}{2} & 0 \end{pmatrix} \end{cases}$$

$\exists$ 5 Stokes matrices of the first linear ODE around $x = \infty$

irreg. sing of Poincaré rank $5/2$

These are t-independent if the fundamental solution satisfies the system of PDEs

The above system must satisfy the compatibility condition

$$\hbar\left(\frac{\partial A}{\delta t} - \frac{\delta B}{\partial x}\right) + [A, B] = 0 \quad \underset{\text{exercise}}{\Longleftrightarrow} \quad \hbar\frac{dq}{dt} = p \quad \& \quad \hbar\frac{dp}{dt} = 6q^2 + t$$

$$\Longleftrightarrow \quad (P_I) \qquad \hbar^2\frac{d^2q}{dt^2} = 6q^2 + t$$

$\rightsquigarrow$ Stokes multiplier around $x = \infty$ are conserved quantities for (P_I)

$$\left\{\begin{array}{l}\text{initial data} \\ \text{for Painlevé eq}\end{array}\right\} \underset{\text{RH Corresp.}}{\longleftrightarrow} \left\{\text{monodromy / Stokes data}\right\}$$

$$\updownarrow$$

$$\left\{\begin{array}{c}\text{flat connection} \\ \frac{\partial Y}{\delta x} = AY\end{array}\right\}$$

Let's apply exact WKB method to study Painlevé eq. (Aoki, Kawai, Takei 96 ~)

Question

$$\hbar\frac{\partial\Psi}{\partial x} = A\Psi$$

Σ : $y^2 = 4x^3 + 2tx + u$ with $u =$ " $\lim_{\hbar\to 0} 2H$ "

$$\Longleftrightarrow \quad L\psi = \left[\hbar^2\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{x-q}\frac{\partial}{\partial x} - \left(4x^3 + 2tx + 2H - \frac{\hbar p}{x-q}\right)\right]\psi = 0$$

where $H = \frac{p^2}{2} - 2q^3 - tq$ ($\Leftarrow$ Hamiltonian for P_I)

What is the classical limit (spectral curve) ?

Does $\lim_{\hbar\to 0} q$ exist ?

Guess from exact WKB philosophie

Monodromy / Stokes data should be computable via Voros symbol (periods on spectral curve)

$\rightsquigarrow$ $\oint_\gamma y\,dx$ should be t-independent for ~~any $\gamma \in H_1(\Sigma;\mathbb{Z})$~~

impossible because t appears in Σ ...

$\rightsquigarrow$ Let us choose a basis $A, B \in H_1(\Sigma;\mathbb{Z})$ and impose $\underbrace{\oint_A y\,dx}_{2\pi i \nu}$ is t-independent.

This determines $u = u(t,\nu)$ at least locally.

$$\left(\frac{\partial u}{\partial t} = 2\frac{\eta_A}{\omega_A} \quad \& \quad \frac{\partial u}{\partial \nu} = \frac{4\pi i}{\omega_A} \qquad \text{where} \quad \omega_A = \oint_A \frac{dx}{y}, \quad \eta_A = -\oint_A \frac{x\,dx}{y} \right)$$

But, how about t-dependence of

$$\begin{cases} B\text{-periods ?} \\ \hbar\text{-corrections of } A\text{-periods ?} \end{cases}$$

Beautifully solved by topological recursion and discrete Fourier transform !

II-2 : τ-function from TR

Let us apply topological recursion to

$$\Sigma : \quad y^2 = 4x^3 + 2tx + u(t,\nu)$$

where $\quad 2\pi i \nu = \oint_A y dx \quad$: t-indep.

- $\omega_{0,1}(z) = y(z)\, dx(z)$

where

$$x = \wp(z), \qquad y = \wp'(z) \qquad z \in \mathbb{C} / \mathbb{Z}\omega_A + \mathbb{Z}\omega_B$$

- $\omega_{0,2}(z_1, z_2) = \left(\wp(z_1 - z_2) + \dfrac{\eta_A}{\omega_A} \right) dz_1 \cdot dz_2$

φ

$$\begin{cases} \cdot \ \text{symmetric} \\ \cdot \ \sim \left(\dfrac{1}{(z_1 - z_2)^2} + \text{hol} \right) dz_1\, dz_2 \quad \text{as } z_1 \to z_2 \\ \cdot \ \oint_{z_1 \in A} \omega_{0,2}(z_1, z_2) = 0 \end{cases}$$

$$\xrightarrow[\text{TR}]{} \quad \omega_{g,n}(z_1, \dots, z_n) \quad \text{and} \quad F_g = F_g(t, \nu)$$

$$F_0 = \frac{t \cdot u}{5} + \frac{\nu}{2} \oint_B y \, dx$$

$$F_1 = -\frac{1}{12} \log\left(\omega_A^6 \cdot D \right), \quad \cdots$$

discriminant

Exercise

Check the equalities $\dfrac{\partial F_0}{\partial t} = \dfrac{u}{2}$, $\dfrac{\partial F_0}{\partial \nu} = \oint_B y \, dx$,

$$\frac{\partial^2 F_0}{\partial \nu^2} = 2\pi i \frac{\omega_B}{\omega_A}$$

- $Z(t, \nu, \hbar) := \exp\left(\sum_{g \geq 0} \hbar^{2g-2} F_g(t, \nu) \right)$

  : perturbative partition function

- $\tilde{\chi}_{\pm}(x, t, \nu, \hbar)$

$$:= \exp\left(\sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^{2g-2+n}}{n!} \int_0^{z(x)} \cdots \int_0^{z(x)} \omega_{g,n}(z_1, \dots, z_n) \right)$$

$n \geq 1 \to n \geq 0$

  : perturbative wave function

$\omega_{g,0} = F_g$

Thm [I 2019]

$\psi_{\pm}(x, t, \nu, \rho, \hbar)$

$$= \frac{\sum_{k \in \mathbb{Z}} e^{2\pi i k \rho/\hbar} \cdot \tilde{\chi}_{\pm}(x, t, \nu + k\hbar, \hbar)}{\sum_{k \in \mathbb{Z}} e^{2\pi i k \rho/\hbar} \cdot Z(t, \nu + k\hbar, \hbar)}$$

are formal solutions of (the first component of) the isomonodromy system with

$$q = -\hbar^2 \frac{d^2}{dt^2} \log \Bigl(\underbrace{\sum_{k \in \mathbb{Z}} e^{2\pi i k \rho/\hbar} \cdot Z(t, \nu + k\hbar, \hbar)}_{=: \tau_I(t, \nu, \rho, \hbar)} \Bigr)$$

That is, τ_I is formal series-valued τ-function

$(\nu, \rho) \longleftrightarrow$ initial conditions for (P_I)

Remarks

(i) $$\mathcal{T}_I(t,v,\rho,\hbar) = Z(t,v,\hbar)\cdot\sum_{m\geq 0}\hbar^m\,\Theta_m(t,v,\rho,\hbar)$$

written by ϑ-function

$$\Theta_0 = \sum_{k\in\mathbb{Z}} e^{2\pi i k\left(\frac{\phi+\rho}{\hbar}\right) + k^2\cdot\pi i\frac{\omega_B}{\omega_A}}, \ \cdots$$

where $$2\pi i\,\phi = \oint_B y\,dx = \frac{\partial F_0}{\partial v}$$

$\because$

$$\frac{Z(v+k\hbar)}{Z(v)} = \exp\left(\frac{1}{\hbar^2}\left(F_0(v+k\hbar) - F_0(v)\right) + \cdots\right)$$

$$= \exp\left(\frac{k}{\hbar}\underbrace{\frac{\partial F_0}{\partial v}}_{=\oint_B y\,dx} + \frac{k^2}{2}\cdot\underbrace{\frac{\partial^2 F_0}{\partial v^2}}_{=2\pi i\frac{\omega_B}{\omega_A}}\right)\left(1+O(\hbar)\right)$$

$\mathcal{T}_I$ is an example of non-perturbative partition function

introduce in [Eynard - Mariño 08] .

(ii) The above construction was generalized to all Painlevé equations by

[Eynard – Garcia-Failde – Marchal – Orantin 19]

(iii) This is closely related to "Kyiv formula"

[Gamayun - Iorgov - Lisovyy 12]

$$\underset{(\hbar=1)}{\tau_{P_{VI}}} = \underbrace{\sum_{k\in\mathbb{Z}} e^{2\pi i k \rho} \cdot \underbrace{C(\nu+k)}_{\substack{\text{product of}\\ \text{Barnes G-function}}} \cdot \underbrace{\mathcal{B}(t,\nu+k)}_{\substack{c=1\ \text{4point}\\ \text{Virasoro conformal block}}}}_{=Z_{\text{Nekrasov}}\ \ N_f=4,\ \varepsilon_1=-\varepsilon_2}$$

Question (c.f., Bonelli's talk)

Can we rigorously relate

$$Z_{TR} \longleftrightarrow \mathcal{B}_{CFT} \longleftrightarrow Z_{\text{Nekrasov}} \ (Z_{AD})$$

including irregular singular cases? perturbative QC $\longleftrightarrow$ BPZ equation

c.f., [Mariño 08], [BKMP 08]

[Kozçaz – Pasquetti – Wyllard 10], [Awata et. al 10], …

[Nagoya 15 ~], … , [Poghosyan - Poghossian 23]

Idea of proof

<u>Thm</u> [I 2019]

$F = \log Z = \sum_{g \geq 0} \hbar^{2g-2} F_g$

$\chi_\pm$ satisfies

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - 2\hbar^2 \frac{\partial}{\partial t} - \underline{\left(4x^3 + 2tx + 2\hbar^2 \frac{\partial F}{\partial t} \right)} \right] \chi = 0$$

$= 4x^3 + 2tx + u + O(\hbar)$

i.e., This is perturbative quantum curve

TR also implies

$$\text{(i)} \quad \oint_{z_i \in A} \omega_{g,n} = \begin{cases} 2\pi i \nu & (g,n) = (0,1) \\ 0 & (g,n) \neq (0,1) \end{cases}$$

$$\rightsquigarrow \tilde{\chi}_\pm \longmapsto e^{\pm 2\pi i \nu / \hbar} \cdot \tilde{\chi}_\pm$$

term-wise A.C along A-cycle

inv. under $\nu \mapsto \nu + \hbar$

$$\rightsquigarrow \psi_\pm \longmapsto e^{\pm 2\pi i \nu / \hbar} \psi_\pm$$

t-independence of "A-Voros period."

(ii) $$\oint_{z_{n+1}\in B} \omega_{g,n+1}(z_1,\dots,z_n,z_{n+1}) = \frac{\partial}{\partial \nu}\Big|_{x(z_i)\,:\,\text{fixed}} \omega_{g,n}(z_1,\dots,z_n)$$ "Variation formula"

$$\rightsquigarrow \quad \tilde{\chi}_{\pm}(t,\nu;\hbar) \longmapsto \tilde{\chi}_{\pm}(t,\nu\pm\hbar;\hbar)$$

term-wise A.C along B-cycle

Excercise

$$\rightsquigarrow \quad \psi_{\pm} \longmapsto e^{\mp 2\pi i \rho A_\hbar}\, \psi_{\pm}$$

t-independence of "B-Voros period"

Using these facts, we can prove that

$\psi_{\pm}$ satisfies the isomonodromy system.

$$\boxed{\text{Summery : non-perturbative quantum curve} = \text{isomonodromy system}}$$

$\boxed{\text{Lecture 4}}$

Previous lecture:

$$y^2 = 4x^3 + 2tx + u \ , \quad \oint_A y\,dx = 2\pi i \nu$$

$$\xrightarrow[\text{TR + dFT}]{} \quad \tau_{\mathrm{I}}(t,\nu,\rho,\hbar) = \sum_{k \in \mathbb{Z}} e^{2\pi i k\rho/\hbar} Z(t, \nu + k\hbar, \hbar)$$

: formal τ-function of (P_{I}) : $\hbar^2 \dfrac{d^2 q}{dt^2} = 6q^2 + t$

Question

- Is τ_{I} (or Z) Borel summable ?
- Can we study its Stoke jump ?

Goal : We will derive a Stokes jump formula for τ_{I} & Z based on the integrability of Painlevé equation (with several conjectural arguments)

II-3 : Conjectures on linear Stokes data

Stokes multipliers of $L_{\mathrm{I}}\psi = 0$ around $x = \infty$

Recall : (P_{I}) $\Leftrightarrow$ isomonodromy deformation of

$$L\psi = \left[\hbar^2 \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{x-q}\frac{\partial}{\partial x} - \left(4x^3 + 2tx + 2H - \frac{\hbar p}{x-q} \right) \right] \psi = 0$$

$$\psi_{\pm} = \sum_{k \in \mathbb{Z}} e^{2\pi i k \rho/\hbar} \, Z(t, \nu + k\hbar, \hbar) \cdot \chi_{\pm}(x, t, \nu + k\hbar; \hbar) \Big/ \tau_I$$

$$\chi_{\pm} = \exp\left(\sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^{2g-2+n}}{n!} \int_0^{z(x)} \cdots \int_0^{z(x)} \omega_{g,n} \right)$$

WKB solution

ψ satisfies PDE (perturbative QC)

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - 2\hbar^2 \frac{\partial}{\partial t} - \left(4x^3 + 2tx + 2\hbar^2 \frac{\partial F}{\partial t} \right) \right] \chi = 0$$

classical lim $\longrightarrow$ $y^2 = 4x^3 + 2tx + u$

<u>Conjecture 1</u> [I 2019]

(i) Borel summability should be <u>controled</u> by the Stokes graph determined by the quadratic differential $(4x^3 + 2tx + u)\, dx^2$ as well as the Schrödinger-type ODE.

i.e., If $\nexists$ saddle conn, then $\chi_{\pm}$ are Borel summable on each Stokes region.

(c.f., [Drukker - Putrov - Mariño 11], ...)

(ii) The Borel sums of $\chi_{\pm}$ are glued by Voros connection formula / path-lifting rule.

(c.f., [Hao - Neitzke 24])

Digression (a class of genus 0 spectral curves)

- $y^2 = \frac{x^2}{4} - \nu \quad (\nu \in \mathbb{C}^*)$: Weber curve

$$\underset{\mathrm{TR}}{\rightsquigarrow} \quad F_g^{\mathrm{Web}} = \frac{B_{2g}}{2g(2g-2)} \cdot \nu^{2-2g} \qquad (g \geq 2)$$

[Harer–Zagier, Penner, Norbury, ...]

$$\mathcal{B}\Big(\sum_{g\geq 2} \hbar^{2g-2} F_g^{\mathrm{Web}}\Big) = \frac{1}{12\zeta} \frac{e^{2\zeta/\nu} + 10 e^{\zeta/\nu} + 1}{(e^{\zeta/\nu} - 1)^2} + \cdots \quad \text{(excercise)}$$

has poles at $\zeta = 2\pi i \nu \cdot k \quad (k \in \mathbb{Z})$

$\therefore \ Z^{\mathrm{Web}}$ is Borel summable iff $\nu \notin i\mathbb{R}$

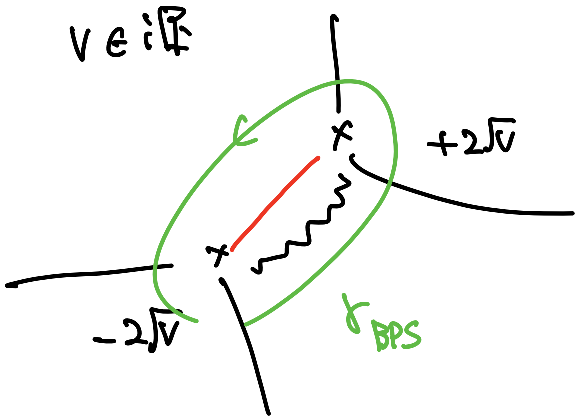
no saddle condition in Stokes graph

$Z(\gamma_{\mathrm{BPS}})$

$= \oint_{\gamma_{\mathrm{BPS}}} y\,dx$

$= \pm 2\pi i \nu$

$\nu \in i\mathbb{R}$



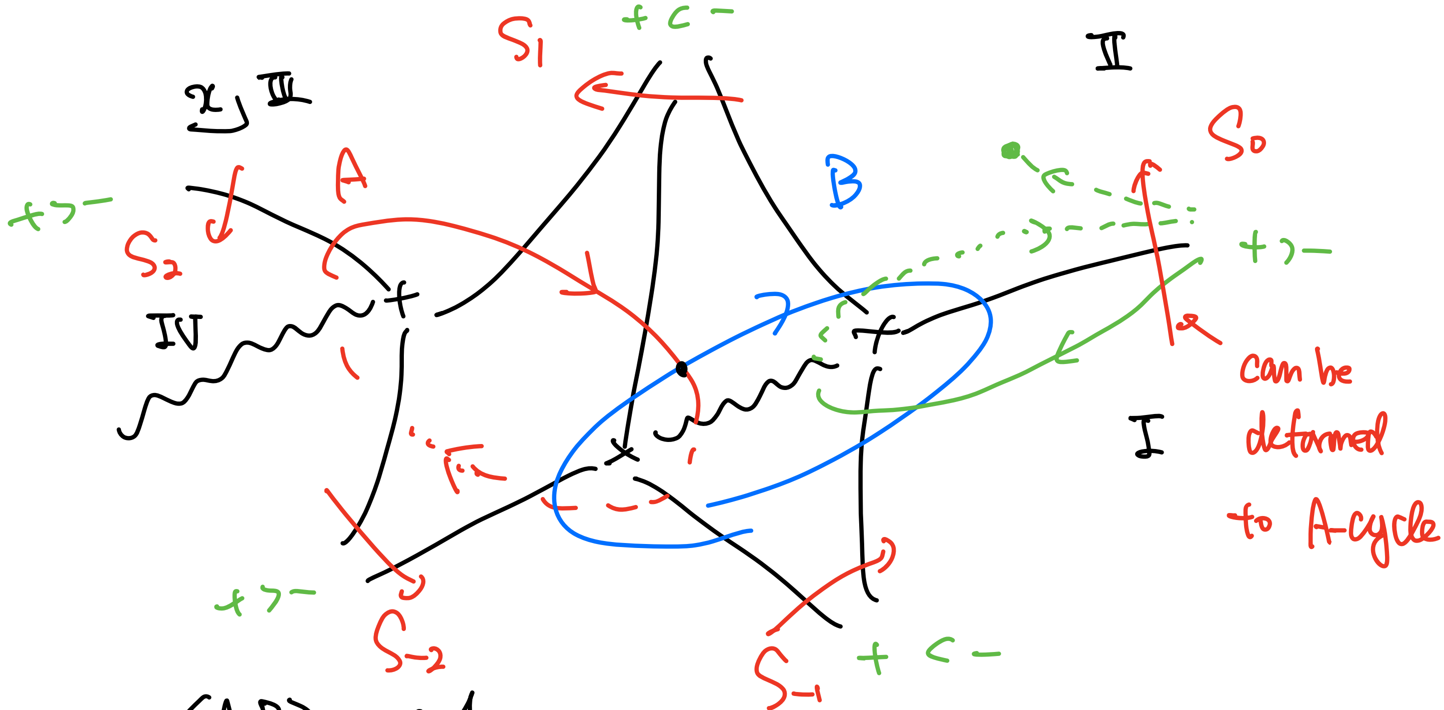
(c.f., I–Kidwai 20, 21)

$\{$Borel sing of $F\} \subset$ period lattice of $y\,dx = \omega_{0,1}$

This is conjectured in general (c.f., [Drukker–Putrov–Mariño 11])

Suppose that the conjecture is true and let us compute the linear Stokes data.

Stokes graph for some t & $u(t,v)$:



$\langle A, B \rangle = +1$

Recall : $\chi_\pm = \exp\left(\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_0^{z(x_1)} \cdots \int_0^{z(x_n)} \omega_{g,n} \right)$: normalized at $x = \infty$

$z = 0 \iff x = \infty$

The Voros formula / path-lifting rule implies

$$\chi_+^{\mathrm{I}} = \chi_+^{\mathrm{II}} + \widehat{\chi}_-^{\mathrm{II}}$$

where $\widehat{\chi}_-$ is obtained by term-wise A.C. along detoured path which can be deformed to A-cycle.

$$\therefore\ \widetilde{\chi}_-^{\mathrm{II}} = i e^{2\pi i \nu/\hbar} \cdot \chi_-^{\mathrm{II}} \quad \text{and}$$

$$\begin{cases} \chi_+^{\mathrm{I}} = \chi_+^{\mathrm{II}} + i \underbrace{e^{2\pi i\nu/\hbar}}_{q\text{ inv under } \nu \mapsto \nu+\hbar} \chi_-^{\mathrm{II}} \\ \chi_-^{\mathrm{I}} = \chi_-^{\mathrm{II}} \end{cases}$$

$$\Big\updownarrow \text{dFT}$$

$$\begin{cases} \psi_+^{\mathrm{I}} = \psi_+^{\mathrm{II}} + i e^{2\pi i\nu/\hbar} \psi_-^{\mathrm{II}} \\ \psi_-^{\mathrm{I}} = \psi_-^{\mathrm{II}} \end{cases}$$

Thus we have the Stokes matrix

$$S_0 = \begin{pmatrix} 1 & 0 \\ i\underbrace{e^{2\pi i\nu/\hbar}}_{s_0} & 1 \end{pmatrix}$$

Next, let us look at S_2:

Voros/path-lifting $\Rightarrow$ $\chi_+^{\mathrm{III}} = \chi_+^{\mathrm{IV}} + \widetilde{\chi}_-^{\mathrm{IV}}$

$\widetilde{\chi}_-(x,t,\nu;\hbar)$ = term-wise A.C. of χ_- along B-cycle

$$= i \cdot \frac{Z(\nu-\hbar;\hbar)}{Z(\nu;\hbar)} \chi_-(x,t,\nu-\hbar;\hbar)$$

$$\underset{\mathrm{dFT}}{\rightsquigarrow}\begin{cases} \psi_+^{\mathrm{III}} = \psi_+^{\mathrm{IV}} + i\, e^{2\pi i p/\hbar}\, \psi_-^{\mathrm{IV}} \\ \psi_-^{\mathrm{III}} = \psi_-^{\mathrm{IV}} \end{cases}$$

$$\therefore \quad S_2 = \begin{pmatrix} 1 & 0 \\ \underbrace{i\, e^{2\pi i p/\hbar}}_{s_2} & 1 \end{pmatrix}$$

By this method, we have the conjectural list of Stokes multipliers s_j :

$$\begin{cases} s_{-2} = i\,(X_B^{-1} - X_A \cdot X_B^{-1}) \\ s_{-1} = i\,(X_A^{-1} - X_A^{-1} \cdot X_B) \\ s_0 = i\, X_A \\ s_1 = i\,(X_A^{-1} - X_A^{-1} X_B + X_B^{-1}) \\ s_2 = i\, X_B \end{cases}$$

where $X_A = e^{2\pi i r/\hbar}$, $X_B = e^{2\pi i p/\hbar}$

"Voros symbols of non-perturbative quantum curve"

↳ derivation was heuristic, but it agrees with several known results.

- cyclic relation $A_k = i\left(1 + A_{k-1} A_{k+1}\right)$

- elliptic asymptotic formula

$$q(t, v, p, \hbar) = \wp\left(\frac{5t}{4\hbar} + \underbrace{\left(\frac{p}{\hbar} + \frac{1}{2}\right)\cdot \omega_A + \left(\frac{v}{\hbar} + \frac{1}{2}\right)\cdot \omega_B}_{=\ \frac{1}{2\pi i}\left(\log(iS_2)\cdot \omega_A + \log(iS_0)\cdot \omega_B\right)} \right) + \text{“}O(\hbar)\text{”}$$

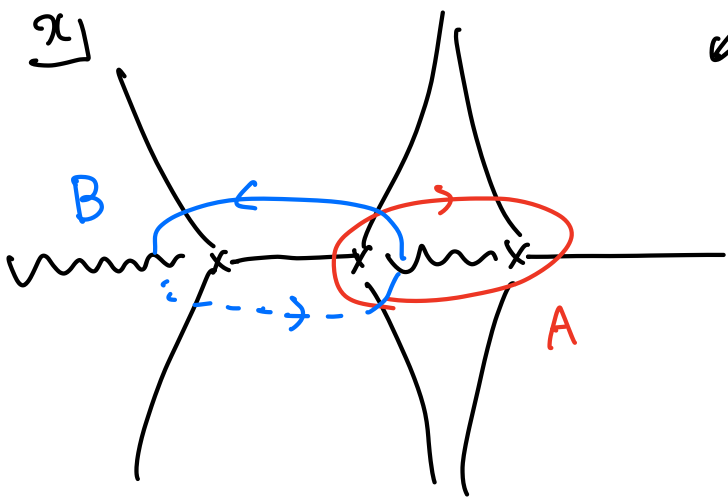
c.f., [Kitaev 89]

Question

Can we make our derivation rigorous math?

III-4: Application to resurgence (further heuristic arguments)

Suppose we have a <u>saddle connection</u> for some t.



e.g.,

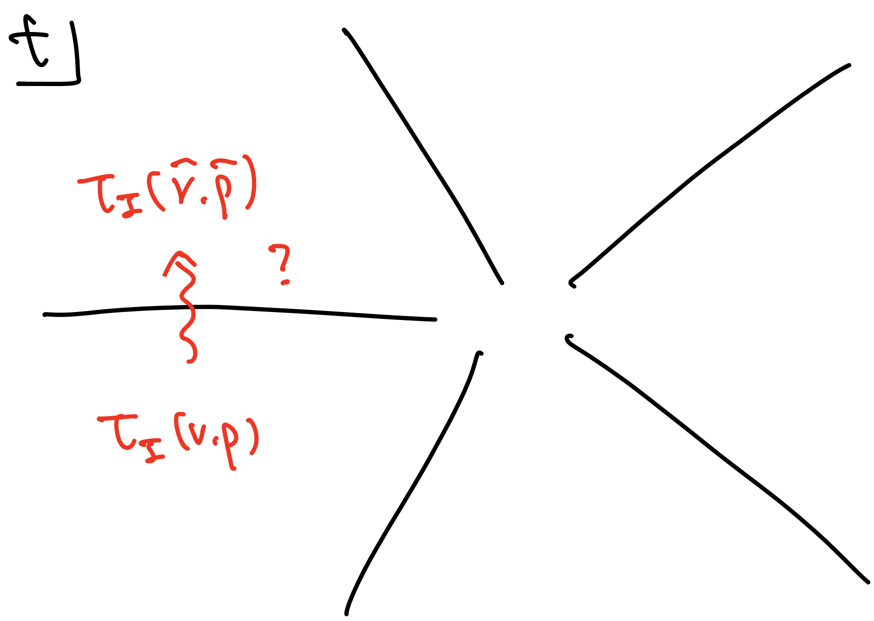
$t \ll 0$

$v \in \mathbb{R}_{>0}$

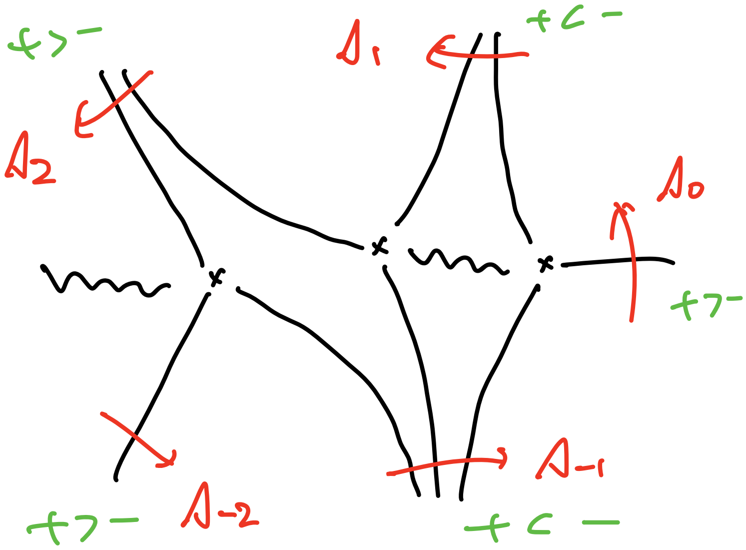
$\hookrightarrow \oint_B y\,dx \in \mathbb{R}$ (expected Borel sing. lie on Laplace contour)

Q : What is the Stokes jump?

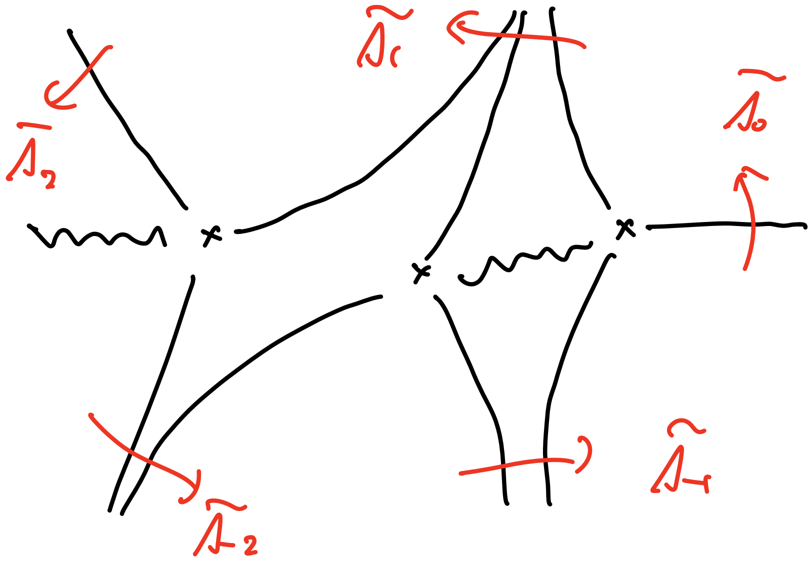
Let us perturb t :



$\mathrm{Re}\,t \ll 0$ & $\mathrm{Im}\,t = -\delta$



$\mathrm{Re}\,t \ll 0$ & $\mathrm{Im}\,t = +\delta$



Previous computation shows

$$\mathcal{A}_{-2} = i\, X_A$$

$$\mathcal{A}_{-1} = i\,(X_A^{-1} - X_A^{-1} X_B^{-1} + X_B^{-1})$$

$$\mathcal{A}_0 = i\, X_B$$

$$\mathcal{A}_1 = i\,(X_B^{-1} - X_A X_B^{-1})$$

$$\mathcal{A}_2 = i\,(X_A^{-1} - X_A^{-1} X_B)$$

$$\tilde{\mathcal{A}}_{-2} = i\,(\tilde{X}_A - \tilde{X}_A \tilde{X}_B)$$

$$\tilde{\mathcal{A}}_{-1} = i\,(\tilde{X}_B^{-1} - \tilde{X}_A^{-1} \tilde{X}_B^{-1})$$

$$\tilde{\mathcal{A}}_0 = i\, \tilde{X}_B$$

$$\tilde{\mathcal{A}}_1 = i\,(\tilde{X}_B^{-1} - \tilde{X}_A \tilde{X}_B^{-1} + \tilde{X}_A)$$

$$\tilde{\mathcal{A}}_2 = i\, \tilde{X}_A^{-1}$$

$$\begin{cases} X_A = e^{2\pi i v/\hbar} \\ X_B = e^{2\pi i p/\hbar} \end{cases}$$

$$\begin{cases} \tilde{X}_A = e^{2\pi i \tilde{v}/\hbar} \\ \tilde{X}_B = e^{2\pi i \tilde{p}/\hbar} \end{cases}$$

Since t is isomonodromic time,

we should have $A_j(\nu,\rho) \overset{!}{=} \widetilde{A}_j(\widetilde{\nu},\widetilde{\rho})$

i.e., $\begin{cases} X_B = \widetilde{X}_B \\ X_A = \widetilde{X}_A(1-\widetilde{X}_B) \end{cases}$

$\rightsquigarrow$ cluster transform / DDP formula / Kontsevich – Soibelman transform

(4d wall-crossing)

Conjecture [I- Mariño 23]

In the above situation, we have

$$\tau_I(t,\nu,\rho,\hbar) \underset{A.C}{\overset{\vartheta}{\rightsquigarrow}} e^{\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{2\pi i\widetilde{\rho}/\hbar}\right)} \cdot \tau_I(t,\widetilde{\nu},\widetilde{\rho},\hbar)$$

where $\widetilde{\nu} = \nu - \frac{\hbar}{2\pi i}\log\left(1-e^{2\pi i\rho/\hbar}\right)$, $\widetilde{\rho} = \rho$

Looking at 0-Fourier mode, we have

$$Z(t,v,\hbar) \underset{\text{A.C}}{\overset{\sigma}{\rightsquigarrow}} \exp\left[\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{-\hbar d_c}\right) - \frac{\hbar d_c}{2\pi i}\log\left(1-e^{-\hbar d_c}\right)\right] Z(t,v,\hbar)$$

$$= \sum_{n=0}^{\infty} Z^{(n)}(t,v,\hbar)$$

[I-Mariño 23]

where $Z^{(0)} = Z(t,v;\hbar)$

$$Z^{(1)} = \left(1 + \frac{\hbar}{2\pi i}\frac{\partial F}{\partial v}(v-\hbar,\hbar)\right)\cdot Z(t,v-\hbar,\hbar)$$

$$\vdots$$

$$\sim Z(v) \times e^{-\frac{1}{\hbar}\oint_B y\,dx}$$

alternative derivation of formulas in

[Gu-Mariño 22] [Gu-Kashani-Poor-Klemm-Mariño 23]

[Mariño's Les Houches lecture]

base on non-perturbative analysis of HAE

(c.f., [Couso-Santamaria-Edelstein-Schiappa-Vonk 14])