Les Houches Lectures on 2D Gravity and Random Matrix Models

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Abstract

(This is a draft(!), improving on L1-L3 and outlining L4. Also, figures and bibliography to come!) These four lectures ~~were~~ (are to be) given at the Les Houches 2024 Summer School on Quantum Geometry, 5th–9th August 2024. The material begins with early motivations for studying 2D quantum gravity: as a route to understanding the path integral definition of string theory, but then moves on to the effective 2D quantum gravity that arises when studying near extreme charged black holes. Both applications emphasize the two primary (and intersecting) interpretations of the random matrix model: As a 't Hooftian means of defining the sum over surfaces through tesselations, and as a Wignerian means of statistically characterising the spectrum of the (holographic dual) Hamiltonian of the theory. Perturbative and non-perturbative aspects are uncovered in detail.

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1 Lecture 1

Before diving into the details of quantizing 2D gravity, it is worth reminding ourselves why we want to do such a thing at all. Of course, one simple reason is that it is a simple enough theory that we might learn useful lessons for quantizing gravity in other dimensions, but there are (at least) two major areas where issues of quantum gravity in higher dimensions already rely on our understanding of quantum gravity in 2D. The first is string theory. A common approach to it involves formulating the dynamics of strings moving in some background in terms of a two dimensional "Polyakov" action for the worldsheet of the string. The spacetime

cooordinates resemble fields in some 2D spacetime, but the dynamics also involves summing over all possible geometries and topologies of the worksheet, which is a form of quantum gravity. The second motivation is the physics of higher dimensional black holes, where (for example) the low temperature physics of Reissner-Nordström black holes reduces to a study of the effective 2D theory that governs the geometry of the near-horizon region.

1.1 Motivations from string theory

Outside of the famous critical dimensions of string theory, the effective scalar coming from Weyl rescaling of the 2D metric does not decouple from the theory, and must be included for consistency. What is typically done is to choose a reference metric $\hat{g}_{ab}$ and write the physical metric as conformal to that: $g_{ab} = \mathrm{e}^{2\phi}\hat{g}_{ab}$, where ϕ is a scalar for which there is a a special 2D "Liouville" conformal field theory action. There's also additional fields (we'll refer to as X generically) with their own CFT action (for example they could be interpreted as additional spacetime coordinates) and of course there are the Fadeev-Popov ghosts from gauge fixing, which have their own CFT. These theories are all coupled together implicitly through the requirement that the total central charge of the theory should vanish. Ignoring the ghost sector henceforth, the problem of Liouville+matter is an important 2D quantum gravity theory with features that will inspire us later, so we should talk about it.

1.1.1 Liouville theory plus matter

The matter+Liouville sector has partition function:

$$Z = \int D\varphi DX \mathrm{e}^{-S_{\text{tot}}} , \tag{1}$$

where S_{tot} includes $S_{\text{L}}(\varphi)$ and $S_{\text{matter}}(X)$, which will be taken to be some conformal field theory representing the matter sector such that

$$c_{\text{L}} + c = 26 , \tag{2}$$

cancelling the -26 coming from the ghost sector. These days the Liouville action is usually written for some $\varphi = \phi/b$ in the following form:

$$S_{\text{L}} = \frac{1}{4\pi}\int d^2z\sqrt{\hat{g}}\left\{\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + Q\hat{R}\varphi + \mu\mathrm{e}^{2b\varphi}\right\} , \tag{3}$$

where the Q term modifies the stress tensor for a scalar to (writing *e.g.* the holomorphic part)

$$T(z) = -\partial_z\varphi\partial_z\varphi + Q\partial_z^2\varphi , \tag{4}$$

with which (in computing the usual OPE) for a vertex operator $\exp(ik\cdot\varphi)$ yields the following (Q-shifted) conformal dimension: $\Delta = -\left(\frac{1}{4}k^2 + \frac{i}{2}Qk\right)$. The last term in the action looks like the insertion of a background involving such a vertex operator with $ik = 2b$, and since conformal invariance requires that $\Delta = 1$ we obtain the condition:

$$Q = b + b^{-1} . \tag{5}$$

The $T(z)T(z)$ OPE yields the central charge $c_L = 1+6Q^2$ for the Liouville sector, and from (2):

$$Q = \sqrt{\frac{25-c}{6}} , \quad \text{with} \quad b = \frac{Q}{2} - \frac{\sqrt{Q^2-4}}{2} = \frac{1}{2}\left[\sqrt{\frac{25-c}{6}} - \sqrt{\frac{1-c}{6}}\right] , \tag{6}$$

where the sign of the root was chosen to match to the classical limit, which is $b \to 0$ or $Q \to \infty$. We will discuss this limit soon.

Recalling that in this action $\hat{g}_{ab}$ is a reference metric and that it is $\hat{g}_{ab}\mathrm{e}^{2b\varphi}$ that is the physical metric, we see that the last term (times $\sqrt{\hat{g}}$) is really the determinant of the physical metric, and hence its integral gives the two dimensional area $A = \int d^2z\sqrt{\hat{g}}\mathrm{e}^{2b\varphi}$. Then μ has the interpretation as a cosmological constant.

1.1.2 KPZ scaling and a critical point

It is natural to wonder about the partition function computed by integrating over surfaces of fixed area, which we can denote as:

$$Z|_A = \int D\varphi DX \mathrm{e}^{-S_{\rm tot}} \delta\left(\int d^2z\sqrt{\hat{g}}\mathrm{e}^{2b\varphi} - A\right), \tag{7}$$

and so the total partition function is

$$Z = \int_0^\infty dA \mathrm{e}^{-\mu A} Z_A\,, \tag{8}$$

a Boltzman factor with an entropic weight. It is interesting to ask what the large A behaviour of Z_A is, and the result (of refs. [?, ?] is that

$$Z|_A \sim A^{(\gamma_{\rm str}^{(0)}-2)\frac{\chi}{2}-1} = A^{\gamma_{\rm str}-3} \tag{9}$$

where $\gamma_{\rm str}^{(0)}$ is often called the (genus zero) string susceptibility, while $\gamma_{\rm str} = 2-(2-\gamma_{\rm str}^{(0)})\chi/2$ is the string susceptibility at genus h, and $\chi = 2-2h$ is the Euler number of the surface. Using the form of the Liouville action, the dependence of $\Gamma_{\rm str}$ on the parameters of the theory can be deduced from a scaling argument. Shifting $\varphi \to \varphi + \rho/2b$, the Liouville term linear in φ shifts the action by $Q\rho\chi/2b$, and within the δ function the integral gets scaled by e^{ρ}. Since $\delta(\alpha x) = \delta(x)/\alpha$, we can write:

$$Z|_A = \mathrm{e}^{-\frac{Q\rho\chi}{2b}-\rho} Z|_{\mathrm{e}^{-\rho}A}\,, \tag{10}$$

whereupon setting $\mathrm{e}^{\rho} = A$ yields:

$$\gamma_{\rm str}^{(0)} = 2-\frac{Q}{b} = \frac{1}{12}\left[(c-1)-\sqrt{(c-1)(c-25)}\right], \quad \text{or} \tag{11}$$

$$\gamma_{\rm str} = 2-\left(\frac{1-h}{12}\right)\left[25-c+\sqrt{(c-1)(c-25)}\right]. \tag{12}$$

(To summarize, we've effectively used the ability to shift φ to rescale area A.) Notice that the asymptotic behaviour (9) allows integral (8) to be done, giving:

$$Z \sim \mu^{2-\gamma_{\rm str}}\,. \tag{13}$$

This is a characteristic behaviour that we'll seek for later. For example for "pure gravty", *i.e.* $c = 0$, on the sphere $g = 0$, $\gamma_{\rm str}^{(0)} = -\frac{1}{2}$, giving $Z \sim \mu^{\frac{5}{2}}$. Notice also that this behaviour also means that the expectation value of the area, $\langle A\rangle = -\partial \ln Z/\partial\mu$, diverges as $\mu \to 0$, which makes sense, and $\gamma_{\rm str}$ is a measure of the rate.

1.1.3 A special classical limit

It's interesting to rescale the Liouville field according to $\varphi = \phi/b$, in which case the action can be wrriten:

$$S_{\rm L} = \frac{1}{4\pi b^2}\int d^2z\sqrt{\hat{g}}\left\{\hat{g}^{ab}\partial_a\phi\,\partial_b\phi + (1+b^2)\hat{R}\phi + b^2\mu e^{2\phi}\right\}\,, \tag{14}$$

with equations of motion:

$$2\nabla^2_{\hat{g}}\phi - \hat{R} = b^2(2\mu e^{2\phi} + \hat{R})\,. \tag{15}$$

Here, the parameter b^2 is playing the role of $\hbar$, and there's an analogue of a classical limit, where $b \to 0$ and hence $Q \to \infty$. Note that the central charges diverge: $c_L \to +\infty$ and $c \to -\infty$. Writing the above in terms of the physical metric $g_{ab} = e^{2\phi}\hat{g}_{ab}$ and using the identity relating 2D Ricci scalars:

$$R(e^{2\phi}\hat{g}) = e^{-2\phi}\hat{g}(R(\hat{g}) - 2\nabla^2_{\hat{g}}\phi)\,, \tag{16}$$

the classical equation of motion is simply

$$R = -2\mu b^2\,, \tag{17}$$

which is Liouville's equation, telling us here (if we hold constant $\mu b^2 = 1$) that the physical metric has constant negative curvature: $R = -2$. This will likely remind you of things (to be) seen in the lectures of Turiaci on JT gravity. Let's have a glance at some of that story.

1.2 Motivations from black holes in $D > 2$.

1.2.1 Reissner-Nordström black hole in $D = 4$

Quick (maybe to be expanded) description of (say) $D = 4$ Reissner-Nordström black holes follows: The extremal (T=0) limit of the metric is $\mathrm{AdS}_2 \times S^2$, with the cosmological constant of AdS_2 and radius of the S^2 set by the charge $\bar{Q}$. There is an $SL(2,\mathbb{R})$ conformal symmetry due to the AdS_2 factor. The black hole has an entropy $S_0 = A/4G_N$, where A is the area of the horizon.

1.2.2 The near-horizon, low-temperature limit

The low-temperature geometry is "nearly" AdS_2 and the sphere's size can fluctuate away from that set by $\bar{Q}$. There is an effective 2D theory of gravity describing the dynamics, with a special coupling to a field Φ that represents the fluctuations of the sphere's radius, resulting in an action that schematically takes the (Euclidean) form:

$$S_{\rm JT} = -\frac{1}{2}\int_M \sqrt{g}\Phi(R+2) - \int_{\partial M}\sqrt{h}\Phi_b(K-1) - S_0\left(\frac{1}{4\pi}\int_M\sqrt{g}R + \frac{1}{2\pi}\int_{\partial M}\sqrt{h}K\right)\,, \tag{18}$$

where R is the Ricci scalar of metric g_{ab}, and for the boundary (∂M) terms, Φ_b is the value of Φ there, h is the induced metric and K is the extrinsic curvature. Euclidean signature has been chosen, such that Euclidean time has period $\beta = 1/T$.

Note that S_0 multiplies $\chi = 2 - 2h - b$, the Euler number of M, where h is the number of handles and b the number of boundaries. (Not to be confused with Liouville parameter!!) This will result in a factor of $e^{\chi S_0}$ in computations using this action. For example, as we will see shortly, for the partition function $Z_{JT}(\beta) = \int D\Phi Dg\,\exp(-S_{\rm JT})$, the leading (disc) order will have a factor e^{S_0} since there $h = 0$ and $b = 1$.

The equation of motion for Φ dynamically sets $R = -2$, and the dynamics reduces to a careful treatment of the boundary dynamics. The latter is a special theory of quantum mechanics where the boundary can fluctuate according to a Schwarzian action, while keeping its length fixed to be β. Turiaci's lectures discuss this in detail, and a key exhibit of this Schwarzian dynamics is the result for the disc partition function:

$$Z_0(\beta) = \frac{e^{S_0}\gamma^{3/2}}{\sqrt{2\pi}}\frac{1}{\beta^{3/2}}e^{\frac{2\pi^2\gamma}{\beta}}, \tag{19}$$

and $\gamma = \Phi_r$, a (renormalized) value of the scalar on the asymptotic boundary where the Schwarzian is defined. (We'll likely set it to $1/2$ later on in these lectures.)

1.2.3 Jackiw-Teitelboim (JT) gravity and the Euclidean GPI

(Probably will say a bit about the general case involving multiple insertions of $Z(\beta)$, and any number of handles.... decomposition into gluing trumpets onto Weil-Petersson volumes. Compute or show a few examples maybe.)

1.3 Performing the Euclidean Path Integral

Let us now step back and look at the problem we've been trying to get to grips with. The core point is that there is some two dimensional gravity theory of g_{ab}, which is largely topological, made interesting by either having a "quantum" deformation (going to non-zero b for Liouville) or by virtue of how it is coupled to an additional sector (dynamical Φ for Jackiw-Teitelboim) or both. The gravitational path integral approach to either story involves summing over all metrics of all topologies. In the case of JT gravity, we were lucky that the dynamics froze the bulk dynamics to be constant curvature metrics, in which case the topological sum over metrics boiled down to enumeration of the properties of the volume of moduli space of hyperbolic metrics, with some needlework to do at the geodesic boundaries in order to connect them to the Schwarzian dynamics living at the asymptotic boundaries where Φ_b lives.

How do we do the sum over metrics and topologies in more general cases? Moreover, what lies beyond the (asymptotic) expansion in the topological expansion parameter? What do we learn about the whole business of defining quantum gravity as a path integral over all geometries and topologies? It's entirely an educated guess/analogy inspired by successes with (non-gravitational) theories, but it isn't guaranteed to be a complete definition. Can we learn in 2D whether it is or not?

To begin to answer some of these questions, let's return to the simplest model we can think of, and start approaching the problem from scratch.

1.3.1 Hermitian random matrix model, tessellations and topology

A possible way to handle the sum over metrics and topologies is to make the problem more manageable by breaking the surface up into pieces. Imagine (see figure XX) using little quadrilaterals ("squares") of some fixed area to build up curved surfaces.

- We can make discrete surfaces of any topology by gluing them together to make something with v vertices, e edges, and n faces (the number of squares). The Euler number of a surface made this way is $\chi = v - e + n$.
- Local positive or negative curvature at some point can be approximated by packing in more or less than four squares meeting.

The partition function representing the sum over all closed surfaces would be of the form:

$$Z_{\text{discrete}} = \sum_{h=0}^{\infty} \nu_B^{2h-2} \sum_{n=0}^{\infty} \mathrm{e}^{-\mu_B n} Z_{h,n} \ , \tag{20}$$

Where all possible topologies are being summed with some weight $\nu_B^{-\chi}$ and area (total number of squares) are being Boltzman weighted with some bare cosmological constant μ_B.

The question is then how to evaluate the $Z_{h,n}$ "entropic" factors? In this simple case of "pure gravity" where there's no coupling to some other sector to contend with (*i.e.,* $c = 0$ in the language of section 1.1.1), $Z_{h,n}$ is simply the number of tesselations at given genus and area.

Here's a way to count these, (following the classic work of Brezin, Itzykson, Parisi, and Zuber '78). Decorate the squares in one of our diagrams drawn above such that there's a point at the centre of each square, and the lines joining them by crossing the edge connecting adjacent squares. Counting all possible such "dual" diagrams is equivalent to counting our tessellations. But these look like Feynman graphs for some theory of quartic interactions. To be precise, consider the (zero-dimensional) "field theory":

$$Z(N,g) = \mathrm{e}^{-E(N,g)} = \int dM \exp\left\{-N\mathrm{Tr}\left(\frac{M^2}{2} + g M^4\right)\right\} , \tag{21}$$

Where M is an $N \times N$ Hermitian matrix and the measure is $dM = \prod_i M_{ii} \prod_{i<j} dM_{ij} dM_{ij}^*$, the Haar measure over matrix elements.

Propagators in this theory are represented by double lines that can be thought of as each propagating an index of the matrix, and the Feynman rules give a factor of $\frac{1}{N}$ for each propagator and gN for each vertex. (See figure XX))

Let's focus on the factors of N that would result form a computation of a diagram. A connected vacuum diagram with P propagators, V vertices and L loops would have a factor $g^V N^{-P} N^V N^L$ (since every closed loop has a free index running around in it that can take any of N values). So this means that the partition function of our theory (for closed diagrams) can be written as (taking the log for closed sector):

$$\log(Z(N,g)) = \sum_{h=0}^{\infty} \left(\frac{1}{N}\right)^{2h-2} \sum_{n=1}^{\infty} g^n Z_{h,n} \ , \tag{22}$$

since by the dual construction $V = n$, $P = e$, and $L = v$ and $\chi = v - e - n = 2 - 2h$. So comparing to equation (20) we can identify that the matrix model has just the right sort of parameters we need for our discrete model, *i.e.,* we have a map $\nu_B \longleftrightarrow 1/N$ and $\mathrm{e}^{-\mu_B} \longleftrightarrow g$.

Our job now is to evaluate $Z_{h,n}$. Let's start by working at leading order, at sphere topology, which is the leading order in the large N limit.

1.3.2 Leading order in large N, the Dyson gas, eigenvalue repulsion

Since the action of our model involves only the trace of powers of M, it is prudent to work in terms of the eigenvalues of M. The trace gives a large $U(N)$ invariance under $M \to UMU^\dagger$, where $U \in U(N)$, and we can "gauge fix" by writing $M = U\Lambda U^\dagger$ where Λ is the diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots \lambda_N\}$. The integral over matrix elements of M now becomes an integral over the N eigenvalues λ_i and the volume of the unitary group, which will appear as an overall factor (which we can drop). The result is

$$Z(N,g) = \int \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left\{-N \sum_i \left(\frac{\lambda_i^2}{2} + g\lambda_i^4\right)\right\} \ . \tag{23}$$

There is a Jacobian $\Delta^2(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2$ appearing as a result of this procedure. $\Delta(\lambda)$ is the Vandermonde determinant. The Jacobian can be computed directly by conputing the determinant of the metric obtained by writing out $(\delta M)^2$ in the decomposition of M into U and Λ, or by using a Fedeev-Popov type procedure, or by reasoning as follows. The change of variables to the eignenvalue basis makes sense for generic λ_i, but it is going to fail whenever any two are identical. (The change of variables can't distinguish between the two, so it is singular there. There's an enhanced symmetry $U(2)$ in this case.) This means the Jacobian J should vanish whenever this happens. This is like the change of variables from rectangular to spherical polars. There $J = r^2 \sin\theta d\theta d\phi dr$. But at $\theta = 0$ and π there is an ambiguity under rotations of x into y, the change of variables is singular, and $J = 0$. So this determines that $J = \prod_{i<j}(\lambda_i - \lambda_j)^\beta$. Dimensional analysis finishes the job. On the one hand, $dM \sim \lambda^{N^2}$, while on the other, $\prod_i d\lambda_i J \sim \lambda^N \lambda^{\beta N(N-1)/2}$, and hence β must be 2.

A suggestive way of writing our eigenvalue problem is as a "Dyson gas":

$$Z(N,g) = \int \prod_i d\lambda_i \exp\left\{-N\sum_i\left(\frac{\lambda_i^2}{2} + g\lambda_i^4\right) + 2\sum_{i<j}\log|\lambda_i - \lambda_j|\right\}, \tag{24}$$

where there are N particles with positions at λ_i, in a potential $NV(\lambda_i) = N(\lambda_i^2/2 + g\lambda_i^4)$ with a logarithmic (1D Coulomb) interparticle repulsion. Let's look at the $N \to \infty$ limit. Intuitively, we should expect a smooth saddle point solution to our system (at least for some range of g) represented by a droplet of eigenvalues formed by the balance between the attraction from the potential dragging them to the origin and the logarithmic repulsion coming from $\sim N$ other elements of the droplet forcing it to spread out.

To find it, replace λ_i by a smooth parameter $\lambda(X)$, where $X = i/N$ runs from 0 to 1, and sums become integrals according to $\frac{1}{N}\sum_i = \int_0^1 dX$. The model is then $Z(N,g) = e^{-N^2 E(g)_{\rm sph}}$, where:

$$E(g)_{\rm sph} = \lim_{N\to\infty}\left\{\int_0^1 dX\left(\frac{\lambda(X)^2}{2} + g\lambda(X)^4\right) + \int_0^1\int_0^1 dXdY\log|\lambda(X) - \lambda(Y)|\right\}, \tag{25}$$

and $\lambda(X)$ is determined by the equations of motion $\delta E(g)_{\rm sph}/\delta\lambda(X) = 0$, which is:

$$\lambda(X) + g4\lambda(X)^3 - 2\mathrm{P}\int_0^1 \frac{dY}{\lambda(X) - \lambda(Y)}, \tag{26}$$

where P denotes the principal part of the integral.

Solving this can be carried out as follows. Introduce a density of eigenvalues $\rho_0(\lambda)$ defined by $dX = \rho_0(\lambda)d\lambda$, and since our potential is even, our solution is going to lie on some symmetric interval that we can parameterize as $(-2a, 2a)$. We should normalize the density according to $\int_{-2a}^{2a} d\lambda\rho_0(\lambda) = 1$. Our stationarity condition can then be written as

$$\frac{V'(\lambda)}{2} = \mathrm{P}\int_{-2a}^{2a} d\mu\frac{\rho(\mu)}{\lambda - \mu}, \tag{27}$$

where a prime denotes a λ-derivative (and μ is a dummy variable, hopefully not to be confused with μ seen earlier).

Extending to the complex λ plane, consider the function

$$F(\lambda) = \int d\mu\frac{\rho_0(\mu)}{\lambda - \mu} \tag{28}$$

with the following properties:

- It is analytic on the plane, with a cut on the interval $(-2a, 2a)$;
- As $|\lambda| \to \infty$, $F(\lambda) \to 1/\lambda$ (following from the normalization condition);
- It is real for λ real outside of the cut;
- It has a discontinuity across the cut:

$$F(\lambda \pm i\epsilon) = \frac{1}{2}V'(\lambda) \mp i\pi\rho_0(\lambda)\,. \tag{29}$$

The above properties fix $F(\lambda)$ to be of the form:

$$F(\lambda) = \frac{1}{2}V'(\lambda) - \frac{P(\lambda)}{2}\sqrt{(\lambda-2a)(\lambda+2a)}\,, \tag{30}$$

where $P(\lambda)$ is quadratic in λ. (As we wil see later, this is a special case of something more general, where when $V(\lambda)$ is of order p, the polynomial is of order $p-2$.) The density is then extracted as:

$$\rho_0(\lambda) = \frac{1}{2\pi}P(\lambda)\sqrt{(4a^2-\lambda^2)}\,. \tag{31}$$

Some experimentation (expand the square root in for large λ) shows that with $P(\lambda) = A\lambda^2 + B\lambda + C$, one gets $A = 4g$, $B = 0$ and $C = 1 + 8ga^2$ (from simply matching to the terms in $V'(\lambda)/2$) and setting the coefficient of λ^{-1} to unity results in the following equation results for a:

$$12ga^4 + a^2 - 1 = 0\,. \tag{32}$$

How do we use this solution? Well, first note that rewriting the integrals in equation (25) as λ integrals (with the aid of our explicit ρ_0 and also equation (26) allows $E(g)_{\text{sph}}$ to be written as:

$$E(g)_{\text{sph}} - E(0)_{\text{sph}} = \frac{1}{24}(a^2-1)(9-a^2)) - \frac{1}{2}\log a^2\,. \tag{33}$$

Consider working perturbatively around $g = 0$. One can write:

$$a^2 = \frac{1}{24g}\left[(1+48g)^{\frac{1}{2}} - 1\right] \tag{34}$$

$$= 1 - 12g + 288g^2 - g^3 + 290304g^4 - 10450944g^5 + \cdots \tag{35}$$

From here one can solve for $E(g)_{\text{sph}}$ perturbatively, getting

$$E(g)_{\text{sph}} - E(0)_{\text{sph}} = 2g - 18g^2 + 288g^3 - 6048g^4 + \frac{746496}{5}g^5 + \cdots\,, \tag{36}$$

and each of these numbers can be associated to the number of ways of drawing a diagram at genus 0 (a planar diagram) with 1, 2, 3, 4, and 5 vertices. See figure XX. Try computing a few and checking! This is precisely the sort of entropic factors $Z_{h,n}$ we sought, for $h = 0$. You might be worried about the alternating signs here, but it is ok. Throw in an overall minus sign to match to the $\log Z(N,g)$ we're really interested in, and then the odd powers of g have minus signs, but this matches the fact that the sign of g in the original matrix model is such that it is really $(-g)^n$ that should multiply diagrams with n vertices (tessellations with n faces).

Well, overall this is all very nice, but it is not the theory we are looking for. Spheres made out of a handful of tesselated squares are not very good approximations to smooth geometry! We need to take a continuum limit.

1.3.3 A critical point and a continuum limit: Double scaling

In fact, we need to make the number of squares large, and we find that regime by going to the edge of the radius of convergence of the small g expansion, where the perturbative treatment begins to break down. This is at $g_c = -\frac{1}{48}$. Following our noses and expanding in $g - g_c$ small, we find that the leading non-trivial behaviour is:

$$N^2 E(g)_{\text{sph}} \simeq N^2 (g_c - g)^{\frac{5}{2}} + \cdots \tag{37}$$

(there are higher order terms that won't survive the scaling limit to be taken shortly), which translates to the following behaviour for the connected diagrams in terms of the bare parameters we wrote previously:

$$\log(Z_{\text{sph}}(N, g)) \simeq \frac{1}{\nu_B^2} (\mu_B - \mu_c)^{\frac{5}{2}} + \cdots \tag{38}$$

So now comes the famous double scaling limit. While taking the large N limit, let's also approach this special point, and correlate the rate at which we take large N with the approach to the critical point. Write (Brezin-Kazakov, Gross-Migdal, Douglas-Shenker):

$$\mu_B - \mu_c = \mu \delta^4 \,, \quad \nu_B \equiv \frac{1}{N} = \hbar \delta^5 \,, \tag{39}$$

where $\delta \to 0$ in the limit, and we are being playful with the notation $\hbar$ for the renormalized topological expansion parameter $1/N$, but this will fit with modern conventions you'll see elsewhere. Think of δ as setting a length scale, like the size of a square in the tessellation. For a given surface of fixed physical size we are using more and more squares to approximate it (as we are making them smaller) and so the continuum is being approached, and we see that our gravity partition function computed by our methods at this order is:

$$Z_{\text{sph}} \simeq \frac{1}{\hbar^2} \mu^{\frac{5}{2}} \,, \tag{40}$$

which is just the behaviour we hoped to see from the KPZ scaling discussion around equation (13).

1.3.4 Living on the Edge

In fact, we could have done this analysis starting with a cubic matrix model. This would correspond to tessellation with triangles. The detailed expressions one gets turn out to be quite different. V' is quadratic now, so $P(\lambda)$ must be linear, for example. The spectral density is not symmetric any more. But still it will turn out that you can tune the coupling in the potential to find the scaling behaviour (40), suggesting that there is some universality going on.

Stepping back, where does this interesting universal behaviour come from? How can we get more of it? The core point is that the *edge* of the spectral density is controlling the behaviour in the double-scaling limit. We shall develop techniques for speaking directly to that in Lecture 2, but for now from this lecture we have the language to state it clearly:

At generic g, the edge of the density is just a square root fall-off. This is already interesting and well-studied in fact, and there are communities of statistical physicists who live at this square root edge and learn all kinds of universal physics from it. They even scale into it, in a way similar to what we do with gravity! For an all-too-brief moment there, our community and those communities ran along in parallel (early to mid-1990s), glancing over at each other, in an interesting bit of history. (I'll give some references later, but you can get a head start by googling the Tracy-Widom distribution for example, and note the date on that paper.)

However, at $g=g_c$, notice that $a^2=2$, and the $P(\lambda)$ we computed above becomes $\frac{1}{12}(8-\lambda^2)$, resulting in the spectral density:

$$\rho_0(\lambda)=\frac{1}{24\pi}(8-\lambda^2)^{\frac{3}{2}}\,. \tag{41}$$

The fall-off has changed to a $3/2$ power at each edge. This is at the root of the change in universality.

The same change in universality can come about with triangles (cubic potentials) since a linear $P(\lambda)$ is enough to change one or the other edge (but not both, like in the symmetric case) from $(\lambda-\lambda_0)^{\frac{1}{2}}$ to $(\lambda-\lambda_0)^{\frac{3}{2}}$ behaviour.

From the point of view of the gravity theory we find in the continuum limit it is saying that there's universal physics that does not care about the details of the tessellation (triangles or squares) which is precisely as it should be!

1.3.5 Multicritical points and double scaling

First a pause for a change of convention. What will follow will fit better if in the example above we send $\lambda\to\sqrt{2}\lambda$, with the result that $V(\lambda)=\lambda^2+g\lambda^4$ and then $g_c=-\frac{1}{12}$. The resulting spectral density is on the interval $(-2,2)$:

$$\rho_0(\lambda)=\frac{P(\lambda)}{2\pi}\sqrt{4-\lambda^2}\,. \tag{42}$$

It's clear what to do next (Kazakov '89). Including higher order potentials, say $V(\lambda)$ of even order $p=2k$, will result in a $P(\lambda)$ of order $p-2$. That allows for tuning parameters to give $P(\lambda)\sim(4-\lambda^2)^{\frac{p}{2}-1}$ and hence a spectral density with edges that fall off as $(\lambda-2)^{k-\frac{1}{2}}$. In fact, one can derive these "multicritical" potentials under just such considerations and they are:

$$V_{2k}(\lambda)=\sum_{m=1}^{k}g_{2m}\lambda^{2m}=\sum_{m=1}^{k}(-1)^{m-1}\frac{k!(m-1)!}{(k-m)!(2m)!}\lambda^{2m}\,, \tag{43}$$

(*e.g.* the familiar $V_4=\lambda^2-\frac{1}{12}\lambda^4$, and $V_6=\frac{3}{2}\lambda^2-\frac{1}{4}\lambda^4+\frac{1}{60}\lambda^6$) and the spectral densities are:

$$\rho_0(\lambda)=\frac{1}{\pi}\frac{(k!)^2}{(2k)!}(4-\lambda^2)^{k-\frac{1}{2}}\,. \tag{44}$$

Playing a bit more will yield that the generalization of (40) that results from this is:

$$Z_{\rm sph}\simeq\frac{1}{\hbar^2}\mu^{2+\frac{1}{k}}\,, \tag{45}$$

where the case we did upstairs was $k=2$. Looking back at (13) suggests that there could exist a sensible continuum theory for every k such that $\gamma_{\rm str}$ on the sphere is:

$$\gamma_{\rm str}^{(0)}=2-\left(2+\frac{1}{k}\right)=-\frac{1}{k}\,. \tag{46}$$

1.3.6 A special family of CFTs

As mentioned above, for the X sector, we're going to focus on some $c\leq 1$ CFTs, the (p,q) minimal models. The integers p and q are mutually prime (and when $|p-q|=1$, the model is unitary). The central charge is:

$$c=1-\frac{6(p-q)^2}{pq}\,. \tag{47}$$

Some examples: The $(3,2)$ model has $c=0$ and is simply the trivial theory, containing only the vacuum, while the $(4,3)$ theory is a celebrity: It has $c=\frac{1}{2}$ and is the critical Ising model. The $(5,2)$ theory is also of great interest in some circles, with central charge $c=-\frac{22}{5}$ it is non-unitary, and is the critical Lee-Yang model, which can be thought of as the Ising model in a background magnetic field tuned to special imaginary values. Finally, the $(1,2)$ model is also interesting. It has $c=-2$ and pertains to a certain topological model once gravity is included, as we'll mention (I hope) later.

It is fun to work out the Liouville parameters Q and b in equation (6) for this family, with the nice results:

$$Q=\frac{(p+q)}{\sqrt{pq}}\,, \qquad b=\sqrt{\frac{q}{p}}\,. \tag{48}$$

The formula derived earlier for the genus zero string susceptibility is $\gamma_{\rm str}^{(0)}\stackrel{?}{=}2-\frac{Q}{b}=1-\frac{p}{q}$, but this must be used *only* when the model is unitary. It is only in such a case that the constant μ in the Liouville sector properly acts as a cosmological operator in theory. If the theory is unitary, one can write $p=k+1$ and $q=k$, in which case

$$\gamma_{\rm str}^{(0)}=-\frac{1}{k}\,. \qquad \text{(unitary series)} \tag{49}$$

This matches the result we got above in equation (46), and initially in the literature it was thought by many for while that this meant that the multicritical points (first suggested as theories involving matter by Kazakov) corresponded to the unitary series of models, but this turned out to be an illusion. (Things began to become clear in work by Staudacher that compared and contrasted the (gravity coupled) Ising and Lee-Yang model in some detail, Brezin-Kazakov-Douglas-Shenker then identified the issue.) For non-unitary theories after "dressing" the CFT operators with Liouville sector, the resulting spectrum of dimensions of operators is such that the (dressed) identity operator, responsible for measuring area, is no longer the lowest dimension operator in the theory. The μ that emerges from the double scaled random matrix model turns out to couple to the lowest dimension operator, and so the formula for the critical exponent needs modification to take into account a term of the form $\sim\mu\mathcal{O}_{\rm min}e^{b_{\rm min}\varphi}$ in the Liouville theory, where $b_{\rm min}$ is computed from requiring that $\Delta=1$ for the dressed operator, as before. The result is a different formula for the critical exponent:

$$\gamma_{\rm str}^{(0)}=-\frac{2}{p+q-2}\,, \tag{50}$$

for which there is another series that fits the bill. Note that the $(1,2)$, $(3,2)$ and $(5,2)$ models are early entries in a special series, which we can write as $(2k-1,2)$ series. They are $k=1,2,3$ cases, and yes *this is* the same k of the previous subsection! We see using (50) that we get

$$\gamma_{\rm str}^{(0)}=-\frac{1}{k}\,, \qquad \text{(non-unitary } (2k-1,2) \text{ series)} \tag{51}$$

which matches the prediction below equation (45). That's pretty nice, I hope you agree.

1.4 Looking ahead

Before moving forward, it is time to develop some more powerful methods for tackling all this, and through which some of the statements made above about edges and universality will become much more straightforward. And that's just the start of what we can do with those methods! They are the orthogonal polynomial methods developed originally in this context in the classic Bessis, Itzykson and Zuber ('80) paper.

2 Lecture 2

2.1 Orthogonal polynomials

Let's start again, writing general even potential $V(\lambda) = \sum_m g_{2m}\lambda^{2m}$, and so:

$$Z(N,g) = \int \prod_i d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2 \exp\left\{-\frac{N}{\gamma}\sum_i V(\lambda_i))\right\} , \tag{52}$$

where on the LHS, g is shorthand for $\{g_{2m}\}$, and a parameter γ has been introduced for later use. We will carry it along with us for a while, and its value will become apparent later.

Let us a define a family of orthogonal polynomials in λ, denoted $P_n(\lambda)=\lambda^n+\cdots$, that are orthogonal with respect to the measure $d\mu(\lambda) = d\lambda e^{-\frac{N}{\gamma}V(\lambda)}$:

$$\int d\lambda e^{-\frac{N}{\gamma}V(\lambda)} P_n(\lambda)P_m(\lambda) = h_n\delta_{nm} , \tag{53}$$

where h_n is a normalization, and $h_0=\int d\lambda e^{-\frac{N}{\gamma}V(\lambda)}$. There is an infinite set of such polynomials $(n,m \in \mathbb{Z}^+)$, but N of them (counting from $n=0$) will be used to define the matrix model. The $P_n(\lambda)$ satisfy a recursion relation:

$$\lambda P_n(\lambda)=P_{n+1}(\lambda)+R_nP_{n-1}(\lambda) , \tag{54}$$

for even $V(\lambda)$. (There can be one more term, but it vanishes for even potentials. We will see it in action later when we look at another important class of random matrix models. The three-term recursion relation is a bedrock of orthogonal polynomial technology that you implicitly use when you use families of orthogonal polynomials, as you have all over physics.) The fact that the recursion relation truncates follows from orthogonality **(I'll insert a proof later)**.

Before going any further it is worth noting that the simplest example, a Gaussian model with potential $V(\lambda) \sim \lambda^2$, the P_n are Hermite polynomials $H_n(\lambda)$ This gives a nice intuition for how physics is described for the whole model in terms of the polynomials: For any particular one of the λ_i, the problem is simply the simple harmonic oscillator, for which the eigensolutions are the Hermite functions $\psi_n(\lambda_i)\sim e^{-\frac{N}{\gamma}\frac{\lambda^2}{2}}H_n(\lambda_i)$ with energies of the form $\sim(n+\frac{1}{2})$. Each of the N problems with coordinate λ_i can be in any of these quantum oscillator states.

In fact, the N different sectors are largely independent: The physics in one sector does not really affect the physics in another. This will mean that the collective description of the physics across the different sectors (a "many body" description) will be that of a *free* system. The logarithmic repulsion that keeps the λ_i's from coinciding can instead be re-interpreted as having prepared many-body states that are fermionic. We won't pursue that interpretation here, but it can be useful.

By definition, the first step of the following is true, but then the second step uses recursion (54) after multiplying by λ to the left, after which orthogonality is used again:

$$h_{n+1} = \int d\mu P_{n+1}\lambda P_n = \int d\mu[P_{n+2}+R_{n+1}P_n]P_n = R_{n+1}h_n , \tag{55}$$

(where the λ dependence is dropped to avoid clutter), resulting in

$$R_n = \frac{h_n}{h_{n-1}} . \tag{56}$$

The $P_n(\lambda)$ can be used to rewrite the matrix integral $Z(N,g_i)$ itself. Since (by taking linear combinations of rows or columns) the Vandermonde determinant can be written in terms of the first N of the $P_n(\lambda)$ as:

$$\Delta = \det\|\lambda_i^{j-1}\|_{i,j=1}^N = \det\|P_{j-i}(\lambda_i)\|_{i,j=1}^N \,, \tag{57}$$

it is easy to rewrite $Z(N,g)$ in terms of properties of the polynomials, by expanding out the determinant factors and using the orthogonality relation. The outcome will simply be a combinatorial factor multiplied by a factor of h_n from each of the N sectors, with result:

$$Z(N,g) = N! \prod_{n=0}^{N-1} h_n \,, \tag{58}$$

but we have another way or writing this given the product relation (56):

$$\begin{aligned} Z(N,g) &= N! h_0^N R_1^{N-1} \cdots R_{N-2}^2 R_{N-1} \\ &= N! h_0^N \prod_{n=1}^{N-1} R_n^{N-n} \\ &= N! h_0^N \exp\left(\sum_{n=1}^{N-1} (N-n) \log R_n\right) \\ &= N! h_0^N \exp\left(N^2 \cdot \frac{1}{N} \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \log R_n\right) . \end{aligned} \tag{59}$$

We will take this expression further in a moment, but the key point is to realize that if we know the R_n recursion coefficients, we can compute the partition function outright.

2.2 Determining the polynomials by recursion

So the next step is to seek relations for the R_n, and in fact they follow from simple identities that the model satisfies by definition. Let us study the action of $d/d\lambda$ on the $P_n(\lambda)$. It is clear that

$$\int d\lambda e^{-\frac{N}{\gamma} V(\lambda)} P_n(\lambda) \frac{d}{d\lambda} P_n(\lambda) = 0 \,, \tag{60}$$

but if the derivative results in two polynomials of the same order being integrated, we get, using (53):

$$\int d\lambda e^{-\frac{N}{\gamma} V(\lambda)} P_{n-1}(\lambda) \frac{d}{d\lambda} P_n(\lambda) = n h_{n-1} \,. \tag{61}$$

Integrating by parts and using (53) again yields:

$$\frac{N}{\gamma} \int d\lambda e^{-\frac{N}{\gamma} V(\lambda)} V'(\lambda) P_n(\lambda) P_{n-1}(\lambda) = n h_{n-1} \,. \tag{62}$$

This gives non-trivial recursion relations for R_n, because the $V'(\lambda)$ insertion can be simplified recursively using the relation (54), followed by use of orthogonality to do the resulting integrals.

For example in the Gaussian case, $V'=\lambda$, and so the left hand side is:

$$\frac{N}{\gamma} \int d\lambda e^{-\frac{N}{\gamma} V(\lambda)} (P_{n+1}(\lambda) + R_n P_{n-1}(\lambda)) P_{n-1}(\lambda) = \frac{N}{\gamma} R_n h_{n-1} \,, \tag{63}$$

and comparing to the right hand side of (63) yields that $R_n = \gamma n/N$, which is the familiar recursion coefficient for the Hermite polynomials, up to the factor of γ/N.

When higher order potentials are used, there'll be more uses of the recursion relations, resulting in more terms, making the problem non-linear. Let's look at the quartic case we studied before, with $V = \lambda^2 + g\lambda^4$, (note the choice $g_2 = 1$ for the quadratic term *vs* the $\frac{1}{2}$ in Lecture 1, so there will be slightly different algebra; also, g_4 is just written as g as before). After a bit of algebra the result is:

$$R_n\,[2 + 4g\,(R_{n-1} + R_n + R_{n+1})] = \frac{\gamma n}{N}\ . \tag{64}$$

We are now ready to make a swift connection to our earlier explorations at large N.

2.2.1 Some operator notation

A quick pause to introduce some notation that will be useful later. Identities (60) and (63) can be rewritten as

$$\Omega_n \equiv \langle n|V'(\lambda)|n\rangle = 0\ , \qquad \widetilde{\Omega}_n \equiv \frac{\gamma n}{N} - \sqrt{R_n}\langle n-1|V'(\lambda)|n\rangle = 0\ , \tag{65}$$

where the compact notation $|n\rangle \equiv \frac{P_n(\lambda)}{\sqrt{h_n}}$ means (Bessis et al 1980, Gross-Migdal 1990) the orthogonality integral (53) rescaled to give a unit normalized inner product $\langle n|m\rangle = \delta_{mn}$. By the way, the first equation gives no useful information for even potentials. The recursion relation (54) in terms of this notation is given by:

$$\lambda|n\rangle = \sqrt{R_{n+1}}|n+1\rangle + \sqrt{R_n}|n-1\rangle\ . \tag{66}$$

(Derived by dividing throughout by $\sqrt{h_n}$ and then using the relation $h_n = h_{n+1}/R_{n+1}$.)

We can even define raising and lowering operators such that

$$\begin{aligned} &\mathcal{A}^\dagger|n\rangle = |n+1\rangle \quad \text{and} \quad \mathcal{A}|n\rangle = |n-1\rangle\ , \\ \text{as well as} \qquad &\langle n|\mathcal{A} = \langle n+1| \quad \text{and} \quad \langle n|\mathcal{A}^\dagger = \langle n-1|\ . \end{aligned} \tag{67}$$

Multiplication by λ inside the integral, combined with using recursion relation (54), is equivalent to the insertion of the operator:

$$\hat{\lambda} \equiv \sqrt{R_{n+1}}\mathcal{A}^\dagger + \sqrt{R_n}\mathcal{A}\ , \tag{68}$$

with the commutation relations

$$\mathcal{A}^\dagger\sqrt{R_n} = \sqrt{R_{n+1}}\,\mathcal{A}^\dagger\ , \text{and} \quad \mathcal{A}\sqrt{R_n} = \sqrt{R_{n-1}}\,\mathcal{A}\,. \tag{69}$$

This seems might seem like overkill, but it turns out to be very useful for doing calculations rather quickly when there are lots of powers of λ being inserted.

2.3 Large N physics

As before, at large N, the index n/N becomes a continuous coordinate X that runs from 0 to 1. The orthogonal polynomials become functions of X, $P_n(\lambda) \to P(X,\lambda)$, and so do the recursion coefficients: $R_n \to R(X)$. A single shift in the index n is a shift by $\epsilon \equiv \frac{1}{N}$. In these terms we can rewrite the equation (64) as:

$$R(X)\,[2 + 4g\,(R(X-\epsilon) + R(X) + R(X+\epsilon))] = \gamma X\ . \tag{70}$$

The leading large N behaviour comes from dropping the ϵ, giving:

$$12gR(X)^2 + 2R(X) - \gamma X = 0\ , \tag{71}$$

which has solution:

$$R(X) = \frac{-1 \pm \sqrt{1+12g\gamma X}}{12g} \,. \tag{72}$$

Note the similarity with, for example, equation (32), if one were to set $\gamma X = 1$. This is not an accident, and we'll return to this later.

What to do with this solution? Well, let's return to our expression (59) for the partition function and write it in terms of continuous large N variables too. Let's take a logarithm to get the connected diagrams. After throwing away some uninteresting additive constants, we have the rather simple expression:

$$\log Z(N, g_i) = N^2 \int_0^1 (1-X) \log R(X) dX \,. \tag{73}$$

Now let's see how to efficiently get the physics we saw before! Using (72) we can expand $R(x)$ in small g to get:

$$R(X) = X\gamma - 12X^2\gamma^2 g + 288X^3\gamma^3 g^2 - 8640X^4\gamma^4 g^3 + 290304X^5\gamma^5 g^4 + O\left(g^5\right) \,, \tag{74}$$

and its logarithm can similarly be expanded, and inserted into the integral (73). Dividing by N^2 and multiplying by a minus sign produces what was called $E_{\text{sph}}(g)$ in Lecture 1, and indeed, carrying this all out gives:

$$E(g)_{\text{sph}} - E(0)_{\text{sph}} = \frac{1}{2}\gamma g - \frac{9}{8}\gamma^2 g^2 + \frac{9}{2}\gamma^3 g^3 - \frac{189}{8}\gamma^4 g^4 + \frac{729}{5}\gamma^5 g^5 + \frac{40095}{56}\gamma^6 g^6 + \cdots \,, \tag{75}$$

which, upon setting $\gamma = 4$ to match Feynman rule conventions, gives our earlier result (36). Hopefully this has convinced you of the power of the methods we're using!

2.4 Critical physics at large N

Let's now look for critical behaviour. There is potentially interesting behaviour for the integral in the neighbourhood of $X = 1$, but there's also interesting behaviour if $R(X)$ goes to 1, which we might suspect is an interesting place given our earlier work on this same problem. How can this happen? Well, if X were to go to unity and *also* if $\gamma \to 1$, then we can see that $R(X) \to R_c = 1$ when $g \to g_c = -\frac{1}{12}$. Aha, this is (in this normalization) our old friend the critical potential for the quartic case! Indeed, this is where the series expansion of (72) diverges.

Let's set $g = g_c$ and $\gamma = 1$. Our equation for $R(X)$ is simply $-R^2 + 2R = \gamma X$ which is

$$1 - (R-1)^2 = \gamma X \quad \Longrightarrow \quad R(x) = 1 + (1-\gamma X)^{\frac{1}{2}} \,. \tag{76}$$

This can be inserted into the integral to give, in the neighbourhood of $\gamma X = 1$:

$$\log Z \sim N^2 (1-\gamma X)^{\frac{5}{2}} \,, \tag{77}$$

where, putting in the scaling $\gamma X = 1 - \mu\delta^4$ and $1/N = \hbar\delta^5$ yields for the connected partition function:

$$Z_{\text{sph}} \simeq \frac{1}{\hbar^2}\mu^{\frac{5}{2}} \,, \tag{78}$$

reproducing the KPZ scaling result just as we saw before in equation (40).

Note that the introduction of γ has been helpful here. Instead of tuning the potential to criticality in a scaled manner, we instead then inserted the critical value g_c of the coupling and kept γ away from 1. Then criticality is approached by sending γ (or the combination

γX) to unity in a scaled way. The two procedures are equivalent and we will adopt the latter henceforth.

It is now easy to state what happens for higher multicritical points. The critical potentials ensure the following form for $R(X)$:

$$R(x) = 1 + (1-\gamma X)^{\frac{1}{k}} \,, \tag{79}$$

and the integral formula and double-scaling (using general formulae (82) below) indeed yields the generalization $Z_{\rm sph} \simeq \frac{1}{\hbar^2}\mu^{2+\frac{1}{k}}$ that we guessed at the end of Lecture 1.

2.5 A useful representation

Let us turn now to the leading spectral density and see what's going on there in this orthogonal polynomial language. There is a very useful integral representation of the leading density in terms of the function $R(X)$:

$$\rho_0(\lambda) = \frac{1}{\pi}\int_0^1 dX \frac{\Theta(4R(X)-\lambda^2)}{\sqrt{4R(X)-\lambda^2}} \,, \tag{80}$$

where the Θ-function is a reminder that only contributions coming from where the square root is real are kept. A derivation may be found in ref. [?], **and I will review it in a later draft**. Let's try to understand it. Some intuition can be obtained by studying the simple Gaussian case, for which $R = X$ (setting $\gamma = 1$ for simplicity), when the formula readily yields the famous Wigner semi-circle law for the $k = 1$ case:

$$\rho_0(\lambda) = \frac{\sqrt{4-\lambda^2}}{2\pi} \,. \tag{81}$$

In this representation it is clear that the the droplet endpoint positions are at $\lambda_c = \pm 2\sqrt{R_c}$, where $R_c \equiv R(X=1)$. This connects the tuning to critical behaviour to the endpoints of the distribution, as we suggested would emerge in Lecture 1. This will become even sharper below.

While we're at it, we can rapidly deduce that the k-generalization of the equation for $R(X)$, with $\gamma = 1$, can quickly give us the leading spectral densities for the k model. Try it by simply using $R(X) = 1 + (1-X)^{\frac{1}{k}}$ in the integral (80) and recover the expression (44).

2.6 Double Scaling in full

As we saw, in the neighbourhood of the critical points, scaled versions of all the key quantities survive to define the continuum physics. The full set of scalings for any k turn out to be:

$$\begin{aligned} &\gamma = 1+\mu\delta^{2k} \,, \quad \gamma X = 1 + x\,\delta^{2k} \quad \text{and so} \quad X = 1+(x-\mu)\delta^{2k} \\ &R(X) = 1 - u(x)\delta^2 \,, \quad \frac{1}{N} = \hbar\delta^{2k+1} \,, \end{aligned} \tag{82}$$

where as before the parameter $\delta \to 0$ in the limit. We have defined a scaled coordinate x which gives a refined probe of the region near $X = 1$. It will be allowed to range over the whole real line, but remember that the power of δ multiplying it means that the whole real line is zoomed out from the infinitessimal neighbourhood of $X = 1$!. (The region $-\infty \le x \le \mu$ will have special significance shortly.) The scaling function $u(x)$ is the scaling of $R(X)$ away from its critical value, in the neighbourhood of the endpoint of the spectral density as we've seen, and indeed $u(x)$ will probe the behaviour of the endpoint in the scaled region.

With these relations, in the limit $\delta \to 0$ the surviving physics for our gravity partition function (73) is:

$$Z = \frac{1}{\hbar^2}\int_{-\infty}^{\mu}(x-\mu)u(x)dx \ , \tag{83}$$

or alternatively

$$u(x) = \hbar^2 \left.\frac{\partial^2 Z}{\partial \mu^2}\right|_{\mu=x} \ . \tag{84}$$

We can also apply these scalings to the leading spectral density of (80). All the physics comes from the neighbourhood of the $X = 1$ limit and as discussed above, this is also the neighbourhood of an end λ_c of the spectral density. So generally the double-scaling limit ought to include scaling λ away from λ_c by a scaled amount:

$$\lambda = \lambda_c - E\delta^2 \ . \tag{85}$$

The power of δ is fixed by the denominator of the integral representation. The expectation should be that $\tilde{\rho}_0(\lambda)d\lambda \to \rho_0(E)dE$, once the scaling $\lambda = \lambda_c - E\delta^2$ is used. A factor δ^{2k-1} results from putting in the rest of the scaling relations, combining to give a factor δ^{2k+1}. To get something finite at large N, a factor of N should be multiplied in here, which gives the finite combination already identified: $\hbar^{-1}$. Thus:

$$\rho_0(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\mu} dx \frac{\Theta(E-u_0(x))}{\sqrt{E-u_0(x)}} \ , \tag{86}$$

where $u_0(x)$ is the leading part of $u(x)$. This is an extremely useful formula that maybe you've seen used in the literature before. It may have seemed like magic when you first saw it. Now you see where it comes from, and what the parameter x and the function $u_0(x)$ really mean.

2.7 String equations and beyond the sphere

The recursion relations for R_n at higher k become a *differential equation* for $u(x)$, which is often (for historical reasons) called a "string equation". (The trivial $k = 1$ case had no such equation, just the exact relation $u(x) = -x$.) Let's derive it for the quartic case $k = 2$.

We must keep all terms in the equation (71), and we Taylor expand them to give:

$$R(X)\left(2 + 4g\left[3R(X) + \epsilon^2\frac{\partial^2 R(X)}{\partial X^2} + \cdots\right]\right) = \gamma X \ , \tag{87}$$

and now setting $g = g_c = -\frac{1}{12}$ and inserting the scaling relations (82) yields that orders δ^0 and δ^2 cancel to zero, and a non-trivial equation appears at order δ^4, which is:

$$-\frac{\hbar^2}{3}\frac{\partial^2 u(x)}{\partial x^2} + u(x)^2 = -x \ . \tag{88}$$

(To be clear: this came from being able to divide everything by δ^4 and then send $\delta \to 0$, discarding an infinite number of terms that are simply non-universal noise, and the equation is the result of asking what was at order δ^4 to vanish, yielding our result.) The "string equation" of equation (88) is in fact the Painlevé I non-linear ODE. In principle, such equations yield both perturbative and non-perturbative information about the model, which was one of the great discoveries of the classic double-scaling limit papers (Brezin-Kazakov,Gross-Migdal,Douglas-Shenker). Let's look at the perturbative expansion:

$$u(x) = \sqrt{-x} - \frac{1}{24}\frac{\hbar^2}{x^2} - \frac{49}{1152}\frac{\hbar^4}{x^{9/2}} + \cdots \tag{89}$$

which yields, after integrating twice according to (84):

$$Z(\mu) = \frac{4}{15}\frac{\mu^{\frac{5}{2}}}{\hbar^2} + \frac{1}{24}\log|\mu| - \frac{7}{1440}\frac{\hbar^2}{\mu^{5/2}} + \cdots \tag{90}$$

corresponding to the sphere, torus, and double torus orders in perturbation theory. (The torus term is precisely what would be computed in the continuum approach as a one-loop vacuum amplitude using string/CFT techniques, the $\frac{1}{24}$ might be familiar as a bosonic vacuum energy in string theory computations.)

The multicritical models that can be obtained from the higher order potentials in general have for their leading behaviour at genus zero $u_0(x) = (-x)^{1/k}$ (for the Gaussian case, $k{=}1$, it is simply $u(x){=}-x$ to all orders.). Indeed, on integrating this leading behaviour for $u_0(x)$ twice we get the $|\mu|^{2+\frac{1}{k}}$ form discussed beneath equation (9), with $\gamma_s = -\frac{1}{k}$, as anticipated already. As an exercise, I strongly recommend working out the full $k = 3$ double scaling example just to see how it works. You'll get the ODE for the Lee-Yang model coupled to gravity equation emerging at order δ^6 (it is δ^{2k} in general).

The ODEs that arise by following the steps from before for higher k can be written as:

$$\mathcal{R} = 0\,, \quad \text{where} \quad \mathcal{R} \equiv \tilde{R}_k[u] + x\,. \tag{91}$$

and the $\tilde{R}_k[u]$ are kth order polynomials in $u(x)$ and its x derivatives normalized such that their $u(x)^k$ term has coefficient unity. They have a pure derivative piece given by $2k-2$ derivatives acting on $u(x)$, and then various mixed non-linear pieces. The first three of these "Gel'fand-Dikii"polynomials are:

$$\begin{aligned} &\tilde{R}_1[u] = u\,, \qquad \tilde{R}_2[u] = u^2 - \frac{1}{3}u''\,, \quad \text{and} \\ &\tilde{R}_3[u] = u^3 - \frac{1}{2}(u')^2 - uu'' + \frac{1}{10}u''''\,, \end{aligned} \tag{92}$$

where here a prime indicates an x-derivative times a factor of $\hbar$. The higher ones can be generated using a recursion relation:

$$\tilde{R}'_{k+1} = \left(\frac{2k+2}{2k+1}\right)\left[\frac{1}{2}u'\tilde{R}_k + u\tilde{R}'_k - \frac{\hbar^2}{4}\tilde{R}'''_k\right], \tag{93}$$

and the requirement that $\tilde{R}_k[u=0] = 0$. Note that

$$\tilde{R}_k[u] = \frac{\Gamma(\frac{1}{2})\Gamma(k+1)}{2\Gamma(k+\frac{1}{2})}R_k[u]\,, \tag{94}$$

where the latter are the polynomials presented in the original Gel'fand-Dikki paper. (To be clear about notation, these aren't the orthogonal polynomial recursion coefficients R_n!)

2.8 A larger family

Note that the structure of the defining equations (65) for R_n is such that it is linear in the potential. This means that we can add together different potentials, and get new models that are a kind of "interpolation" among the multicritical models. It is natural to think of the $V^{(k)}(\lambda)$ as a kind of basis. Care must be taken on two counts. The first is that we have to make sure to include extra factors of δ^2 for the physics of different values of k to survive the continuum limit. The answer is simple to work out: If the physics is to survive at order $2k$, then adding in $V^{(m)}(\lambda)$ for $m < k$ has to have $\delta^{2(k-m)}$ as a prefactor.

The second is a bit more subtle. In what we saw with $k = 1$ and $k = 2$ and maybe also observed in your practice with other examples, the scaling ansatz (82) is such that there is a cancellation of a $\mathbf{1}$ on the LHS (from the critical value of γX) against one that comes up on the RHS (from R_c). If there are multiple other potentials in the problem, they will introduce additional $\mathbf{1}$s on the RHS that won't cancel. This is easily solved by adding in $-V^{(m+1)}(\lambda)$ with each addition of $V^{(m)}(\lambda)$, since all terms from that will vanish with higher powers of δ *except* the needed $-\mathbf{1}$.

Defining numerical coefficients t_m, the end result is then an interpolating potential

$$V(\lambda) = \sum_{m=1}^{k} t_m \delta^{2(k-m)} \left[V^{(m)}(\lambda) - V^{(m+1)}(\lambda) \right] \tag{95}$$

The resulting more general string equation is:

$$\mathcal{R} = 0 \,, \quad \text{where} \quad \mathcal{R} \equiv \sum_{m=1}^{k} t_m \tilde{R}_m[u] + x \,. \tag{96}$$

These will be very useful later.

2.8.1 The underlying integrable system - the KdV equation

Some of you might recognize some of the objects lurking here as familiar from a particular integrable system, the KdV hierarchy. This is not an accident. The larger string equation (96) we just deduced now says that $u(x)$ is a function not just of x but also all the t_m: $u(x, \{t_m\})$. As they change it should change. This makes sense. Imagine that one had just the equation such that all $t_m = 0$ except t_2 and $t_3 = 1$. Now let t_2 start out vanishingly small and then grow. Initially, the model is gravitating Lee-Yang, while for large t_2, the system is pure gravity, which has quite different behaviour for $u(x)$. Intuitively, this is just the flow between conformal field theories triggered by turning on operators.

What describes the intermediate evolution $u(x, t_2)$? It is an integrable flow equation, the KdV equation (describing soliton solutions in integrable systems, in fact), which for any t_m is:

$$\frac{\partial u}{\partial t_m} \sim \tilde{R}'_{m+1}[u] \,. \tag{97}$$

(with a proportionality constant for my conventions that **I should check.**)

2.8.2 Something about conformal basis *vs* "KdV" basis

Should probably say something about conformal basis *vs* "KdV" basis for the t_k.

...Will add later.

2.9 Looking Ahead

We've achieved one major goal, which is to define to all orders in genus expansion (and possibly beyond?) Liouville gravity coupled to (a class of) matter. This is a big deal. We are also very close to begin understanding JT gravity using these same tools! We also have to talk about non-perturbative physics now that we have the tools to extract it. We have a lot to do!

3 Lecture 3

3.1 A useful quantum mechanics emerges

Returning to the main story, it is useful to look more closely at the organization of the physics the orthogonal polynomials provide. The normalized polynomials, with an additional factor from the measure absorbed into them, $\mathbf{e}^{-\frac{N}{\tau}V(\lambda)/2}P_n(\lambda)/\sqrt{h_n}$, will become a function denoted $\psi(x,E)$ in the double-scaling limit, about which more will be said shortly. The basis denoted previously $|n\rangle$ becomes $|X\rangle$ in the limit. An important question to ask is what becomes, in the limit, of the operation of multiplying the orthogonal polynomials by λ. This is interesting and important to work out. Preparing the recursion operator (54) for the large N limit we have:

$$\lambda|X\rangle=\sqrt{R(X+\epsilon)}|X+\epsilon\rangle+\sqrt{R(X)}|X-\epsilon\rangle\ , \tag{98}$$

which can be written as an operator $\hat{\lambda}$ on the X dependence in the following way:

$$\hat{\lambda}=\sqrt{R(X+\epsilon)}\exp\left\{\epsilon\frac{\partial}{\partial X}\right\}+\sqrt{R(X)}\exp\left\{-\epsilon\frac{\partial}{\partial X}\right\} \tag{99}$$

Taylor expanding and substituting the scaling relations (82) and (85) gives $\hat{\lambda}=2-\mathcal{H}\delta^2+\cdots$, where:

$$\mathcal{H}=-\hbar^2\frac{\partial^2}{\partial x^2}+u(x)\ , \tag{100}$$

is the operator whose eigenvalue is the scaled energy E. The scaled wavefunctions in the limit are denoted $\psi(x,E)$ and are thus the eigenfunctions of $\mathcal{H}$:

$$\mathcal{H}\psi(E,x)=E\psi(E,x)\ , \tag{101}$$

With a known $u(x)$, (*that must extend over all of* x–we'll discuss this later) this then fully defines a quantum mechanics problem. This quantum mechanical system naturally provides the tools with which to excavate the *fully non-perturbative* physics. We will do a simple exactly solvable model soon to see how this all works.

3.2 Case study: the Airy Model

Let's take a step back and go to the simplest model we've seen so far, the Gaussian case ($k=1$ in teh langauge of the last two lectures). The leading spectral density is:

$$\rho_0(\lambda)=\frac{\sqrt{4-\lambda^2}}{2\pi}\ , \tag{102}$$

the famous Wigner semi-circle law. I cannot overemphasize how important it is to have a go at generating your own example "experimentally". I explain how I sampled an ensemble of 100×100 Hermitian matrices, randomly generated on a computer using Gaussian probability. See figure 1, and footnote 1 describes how to make it.[1]

Let's now scale into the $\lambda_c=2$ end by writing $\lambda=2-E\delta^2/N^{2/3}$, where $\delta\to 0$ as $N\to\infty$. This unpacks the details of the spectral density in the neighbourhood of that end. It is going

[1] A computational aside: This came from writing a MATLAB loop that, in each iteration, generated a sample 100×100 Hermitian matrix `M=(C+ctranspose(C))/2` with Gaussian probability through random complex matrix `C=randn(N)+1i*randn(N)` in MATLAB where `N=100`. Then the eigenvalues were gathered, using `eigs(M)`, and dumped into a list (simply a MATLAB vector; call it `alleigs` for argument's sake). This was done 10,000 times using the loop, taking about 3s. Then every element in `alleigs` was divided by `sqrt(N)`. A simple histogramming command (*e.g.,* `histogram(alleigs)`) generated the semi-circle.

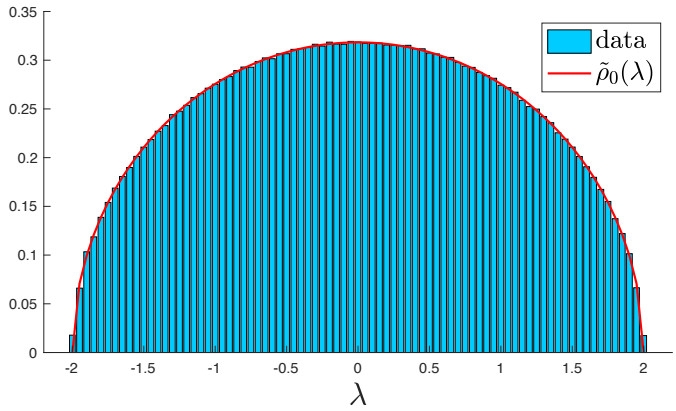

Figure 1: Wigner's semi-circle law, using 10,000 samples of **100×100** Hermitian matrices.

to be small compared to the "bulk" quantity, so writing $\rho_0(\lambda)d\lambda = C\rho_0(E)dE$, where C scales inversely with N, the finite piece $\rho_0(E)$ can be extracted. It is:

$$\rho_0(E) = (\pi\hbar)^{-1}E^{\frac{1}{2}} \,, \tag{103}$$

where it turned out that $C{=}N^{-2}$, and recall that the combination $\hbar = 1/(N\delta^3)$ is held fixed as $N \to \infty$. It is our "renormalized" $1/N$ expansion parameter. (This scaling can be done on your eigenvalue data in the computer code of footnote 1, and will be discussed later.)

3.3 Using the quantum mechanics toolbox: Airy model case study

The quantum mechanics' spectral problem, with wavefunctions $\psi(E,x)$ and energies E:

$$\mathcal{H}\psi(E,x) = E\psi(E,x) \,, \tag{104}$$

(where $\mathcal{H}$ is given in equation (100)) is now the focus. A important note of caution: This quantum mechanics *should not be confused with being dual to the gravity theory.* It is simply an efficient organizing tool for describing the physics of the Dyson gas. Strange operations will be done with it that you would not do with an ordinary QM, and that because it is an auxiliary theory.

Notice that energy E of the quantum mechanical system is actually *position* in the (scaled) Dyson gas problem. On the other hand recall that the position, x, in the quantum mechanics problem began life as the index of the orthogonal polynomials, which correlates with energy excitation of an oscillator problem. This means that x labels an *energy excitation* of a particle at a postion labelled by E in the Dyson gas.

The Gaussian case is illustrative. Recall that the $\psi_n(\lambda) = \mathrm{e}^{-NV(\lambda)/2\gamma}P_n(\lambda)/\sqrt{h_n}$ are Hermite functions, the classic harmonic oscillator wavefunctions. They have energy $\left(n+\frac{1}{2}\right)$ times a constant, confirming that x (the piece of n that survives the double scaled limit) in fact labels an energy. This generalizes to non-Gaussian cases too, as the index n sets the highest power λ^n in the polynomial factor of the wavefunction, controlling the number of nodes it has, and so it (and hence x) remains a good energy coordinate of the Dyson gas.

3.3.1 The Wavefunctions

In the double-scaling limit, for the Gaussian case, $u(x){=}{-}x$ exactly as mentioned, and hence the wavefunctions are simply Airy functions, written as:

$$\psi(E,x){=}\hbar^{-\frac{2}{3}}\mathrm{Ai}[-\hbar^{-\frac{2}{3}}(E+x)] \,, \tag{105}$$

where the normalization is chosen in a way that will match the leading result for the spectral density, as will be shown presently. (As far as I know, it was Moore (1990) who first showed that these arise from a direct scaling limit on the Hermite function (in this context).) In this special case (following from the form of $\mathcal{H}$), the objects E and x come in the combination $(E+x)$ everywhere, resulting in an accidental symmetry on exchanging them, but in general this will not be the case.

3.3.2 Distinguishing partition functions! (And Wigner waves at us with a grin.)

Remember that we had the gravity partition function in Lectures 1 and 2, which we had to distinguish from the random matrix model partition function? Well another partition function is on its way - one that is central to the modern gravity discussion. The generating function (83) for the connected closed 2D universes that occupied our attention so much in the last two lectures will lose the spotlight for a while.

We instead turn to something that has an interpretation as the double-scaled version of the "loop operator" $\langle \frac{1}{N}\mathbf{Tr}[\mathbf{e}^{\ell M}]\rangle$, (where here **Tr** means the matrix trace) which makes a loop of fixed length ℓ in the surface. (The loop interpretation can be understood by looking at the large L limit of the matrix operator $\mathbf{Tr}[M^L]$. Thought of as a vertex, it generates (after going from vertices to tessellations in the t'Hooftian way) a large loop of length L, in units of the lattice spacing. Taking that length large as the lattice spacing goes to zero will make finite loops in the continuum limit.) (See Gross-Migdal, and Banks et. al., "Macroscopic and Microscopic loops..." (1990), and also earlier papers.)

For us, the loop length of interest in the continuum limit will be denoted β, and the the matrix whose eigenvalues we are studying in the DSL will be called H, and the loop operator expectation value will be denoted $\langle \mathbf{Tr}\, \mathbf{e}^{-\beta H}\rangle$. In the appropriate (double-scaled) matrix model to come, this is identified with the JT gravity Euclidean partition function $Z(\beta)$ where there is (as you know from Turiaci's lectures) an asymptotic boundary of length β. In the orthogonal polynomial language developed in the previous subsection it is computed as follows:

$$\langle \mathbf{Tr}\, \mathbf{e}^{-\beta H}\rangle = \int_{-\infty}^{\mu} dx\, \langle x|\mathbf{e}^{-\beta \mathcal{H}}|x\rangle \stackrel{?}{\equiv} Z(\beta)\,, \tag{106}$$

Let's stop for a moment, because I need to do a little rant: I cannot emphasize strongly enough that this is an *average partition function*, $\langle \mathbf{Tr}\, \mathbf{e}^{-\beta H}\rangle$, where H is a matrix drawn from the (double-scaled) ensemble, right at the outset.

Wait, what ensemble? The way we've been studying the matrix model so far has been very much in the style of the string and conformal field theory approach. We took the (middle period) historical approach that the random matrix model partition function $Z(N,g)$ that we wrote down in equation (21) or (52) was a toy quantum field theory whose job it was to, using t'Hooft's ribbon diagrammatic large N toolbox, construct a tessellation of all geometries and topologies and sum them us for us. But that partition function is *also* the basic normalizer of a statistical problem: Draw a matrix M at random with probability:

$$p(M) = \frac{\mathbf{e}^{-\frac{N}{\gamma}V(M)}}{Z(N,g)}\,, \tag{107}$$

where we divide by the matrix model partition function $Z(N,g)$ to make sure our probabilities add up to unity. This is what Wigner brought random matrix technology into physics for (in the context of studying the properties of large nuclei), and the M's were sample Hamiltonians.

The loop operator (106) has not a shred of mystery about it, in that light. You draw an H from the ensemble and compute its spectrum, $\{E_i\}$. You can then, for inverse temperature set

by β, compute the partition function $Z = \sum_i e^{-\beta E_i}$. But don't stop there, draw another H and compute again, repeatedly, and compute the average (across the ensemble) of those numbers you computed each time. That is precisely what the macroscopic loop is. In equation (106), I've tentatively identified it as the gravity path integral, writing it without an averaging symbol, and the holographic expectation might be that H is the Hamiltonian of some dual 1D theory equivalent to the gravity theory. But once we look at multi-point correlators of $Z(\beta)$ (on the gravity side), it is clear that *if you interpret the path integral as an instruction to include all topologies*, then the results are better interpreted as some sort of average. Identifying the average as a computation within a matrix model (as the results urge) then tells you that even the one-point function (the gravity partition function $Z(\beta)$ - defined with the GPI) is an average. So we really should write it as $\langle Z(\beta)\rangle$. This probably fits with the fact that even for the one point function we include a sum over topologies. So let's do that henceforth:

$$\langle Z(\beta)\rangle = \int_{-\infty}^{\mu} dx\, \langle x|e^{-\beta\mathcal{H}}|x\rangle \,, \tag{108}$$

Perhaps the lesson is that the quantum gravity path integral, defined to include different topologies, *must* boil down to some kind of average, and in two dimensions we learn that the average is an ensemble average. Not all averages are ensemble averages, and not all gravity is in two dimensions, so maybe in other dimensions summing over topologies involves other kinds of averaging.

*Any*way...(!), equation (108) is the way the loop expectation (that is the gravity path integral) is computed in the double-scaled technology. I'll show you how it shortly. The integral in equation (108) is a sort of projected trace (in an x-basis) of the exponentiated Hamiltonian, which it is useful to write for future reference as:

$$\langle Z(\beta)\rangle = \mathbf{Tr}[\mathcal{P}e^{-\beta\mathcal{H}}] \quad \text{where} \quad \mathcal{P} \equiv \int_{-\infty}^{\mu} dx\, |x\rangle\langle x| \,, \tag{109}$$

which is a tad notationally clumsy, but $\mathbf{Tr}[\mathcal{O}]$ really means to make the sandwich $\langle x|\mathcal{O}|x\rangle$ and integrate over x, but the projector says only go up to $x = \mu$. This upper limit boils down to, you'll recall, the fact that we're using only N of the orthogonal polynomials, and so the index n only went up to N. This was X only going up to 1, and this translates into x only going up to μ in the double scaling limit. Now, exponentiating $\mathcal{H}$ and taking this trace are not intuitively obvious things to do from the perspective of the simple quantum mechanics (104). However, a few more steps show that the form is quite natural for the problem in hand. Putting a complete set of energy eigenstates $\int d\psi|\psi\rangle\langle\psi| \equiv \int dE|\psi_E\rangle\langle\psi_E| \equiv \mathbf{1}$ into equation (108) and using equation (104) with $\psi(E,x) \equiv \langle\psi_E|x\rangle$ gives the partition function as the Laplace transform:

$$\langle Z(\beta)\rangle = \int dE\rho(E)e^{-\beta E} \,, \tag{110}$$

of the spectral density:

$$\rho(E) = \int_{-\infty}^{\mu} |\psi(E,x)|^2 dx \,. \tag{111}$$

This is a fully non-perturbative expression. If we know $u(x)$ completely, and can compute the wavefunctions $\psi(x,E)$ completely, then this $\rho(E)$ is the whole answer.

What is the meaning of this? Well, remember the remarks about the Dyson gas. There, E is a position, and x is an energy. So at "position" E I can ask what is the amount of excitation turned on, and compute that by summing up the squared wavefunction over all occupied energies, up to some max value μ.

But how is this connected to the expression we had for the (leading) spectral density? Well, take the wavefunction (105) and use its WKB approximation:

$$\psi(E,x) \simeq \frac{1}{\sqrt{\pi\hbar}} \frac{1}{(E+x)^{\frac{1}{4}}} \cos\left(\frac{1}{\hbar}\frac{2}{3}(E+x)^{\frac{3}{2}} - \frac{\pi}{4}\right) , \tag{112}$$

which is really a special case of the more general expression:

$$\psi(E,x) \simeq \frac{1}{\sqrt{\pi\hbar}} \frac{1}{[E-u_0(x)]^{\frac{1}{4}}} \cos\left(\frac{1}{\hbar}\int^x \sqrt{E-u_0(x')}dx' - \frac{\pi}{4}\right) , \tag{113}$$

where $u_0(x)$ is the leading (classical) part of $u(x)$. For the Airy model $u_0(x) = -x$ and is as we've seen happens to be the exact answer. Using this WKB form in the density integral (111) gives the leading classical expression for the leading part of $\rho(E)$, which is what we had before:

$$\rho_0(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\mu} \frac{1}{\sqrt{E-u_0(x)}}dx \ . \tag{114}$$

There is an additional factor of $\frac{1}{2}$ coming from the fact that, for large enough E, the frequency of the oscillations due to the cosine are fast enough to stand being averaged over.

Now let's do the whole problem. Inserting the full expression for the wavefunction (105) and doing the integral (111) (put μ=0 for this case it turns out - not doing so just shifts the origin by μ due to translation invariance) gives the full non-perturbative density as:

$$\rho(E) = \hbar^{-\frac{2}{3}}\left[\mathrm{Ai}'(\zeta)^2 - \zeta\,\mathrm{Ai}(\zeta)^2\right] , \tag{115}$$

with $\zeta \equiv -\hbar^{-\frac{2}{3}}E$, where $f' \equiv \partial f/\partial\zeta$. Note that its support includes $E<0$. In contrast, taking large E gives the classical (disc order) result obtained by just zooming into the end of the Wigner semi-circle, which is supported only on $E\geq 0$. The function is plotted in figure 2. It is a very good exercise to actually write the few lines of code needed to generate random Gaussian matrices and histogram the data. You can even scale into the edge as shown here, and overlay it on the exact result. (I've done finite size matrices here, so there's a bit of shortfall from the $N\to\infty$ exact result.) See the next footnote for a bit on how to do this scaling into the edge numerically.[2] The non-perturbative undulations visible at lower energy have additional microscopic *statistical* information, hardly discussed in the gravity literature, but is nevertheless very important as a model of quantum gravity microstates. I hope to get to it later.

Actually, the Laplace transform of the Airy $\rho(E)$ given in (115) can be computed exactly:

$$\langle Z(\beta)\rangle = \int_{-\infty}^{+\infty} \rho(E)e^{-\beta E}dE = \frac{e^{\frac{\hbar^2}{12}\beta^3}}{2\pi^{1/2}\hbar\beta^{3/2}} \ . \qquad \textbf{(Airy)} \tag{116}$$

Well, you can actually plot this and compare it to the average partition function that I encouraged you to plot earlier. I think you'll be pleasantly surprised.

[2] You might still have your MATLAB window open, this is a good point to return to footnote 1, and re-run the code but now with the shift and scaling given just before equation (103), *i.e,* if `e` is a list of raw matrix eigenvalues, accumulate `E=N^(1/6)*(2*sqrt(N)+e)` over the loop. On histogramming the new data, the bins line up with the curve (115), for some way, although it eventually begin to fall short due to the finiteness of N. This is also shown in figure 2.

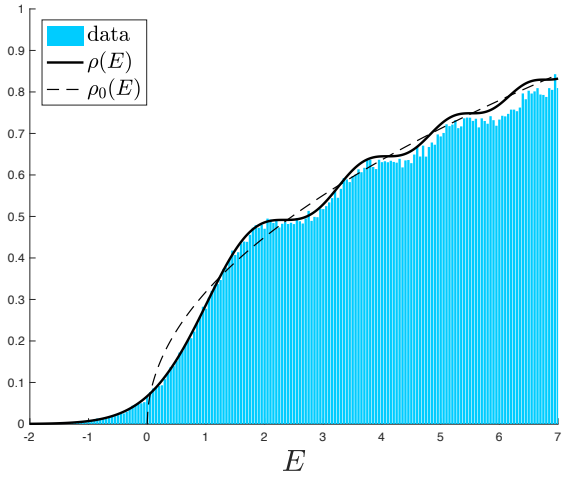

Figure 2: The spectral density $\rho(E)$ for the Airy model (solid line). The rising dashed line is the leading result $\rho_0(E)=\sqrt{E}/(\hbar\pi)$. Here, $\hbar=1$. Also shown are the histograms of data obtained from 100K scaled energy samples of 100×100 Hermitian matrices randomly generated with Gaussian probability. The non-perturbative undulations at lower energy hide additional microscopic information, to be discussed later.

This is actually a (non-perturbative) analogue of what you've seen for JT gravity in Turiaci's Lectures. The more direct analogue would be to transform the leading part, (103), which yields:

$$\langle Z_0(\beta)\rangle = \int_0^{+\infty} \rho_0(E) e^{-\beta E} dE = \frac{1}{2\sqrt{\pi}\,\hbar\beta^{\frac{3}{2}}} \,. \tag{117}$$

Notice the $\beta^{-\frac{3}{2}}$, which is also present in JT. The exponential part is missing however. That's interesting. The former factor came from the one-loop computation that led to a count of zero-modes from broken $SL(2,\mathbb{R})$. That's still happening here. The missing parts are actually the leading classical stuff coming from evaluating the JT gravity action. It perhaps makes sense that isn't here given the topological nature of the model.

3.4 The spectral form factor

Another object you saw in the lectures of Turiaci was the spectral form factor. It seems reasonable to talk about it here, and it will also allow me to show you how a two-point correlator computation works.

The spectral form factor is a two-point function of the partition function whose late time behaviour is a useful diagnostic of the properties of the spectrum of the theory:

$$Z(\beta+it)Z(\beta-it) = \sum_{j,k} \mathrm{e}^{-\beta(E_j+E_k)} \mathrm{e}^{it(E_j-E_k)} \,, \tag{118}$$

for a given energy spectrum $\{E_k\}$. In particular, at late times, fluctuations begin to dominate the quantity, the nature of which depend upon the details of the low energy spectrum. The random matrix model readily yields the ensemble average of this quantity, which smooths out the fluctuations. The result is naturally decomposed into the sum of a disconnected piece and a connected piece:

$$\langle Z(\beta+it)Z(\beta-it)\rangle = \langle Z(\beta+it)\rangle\langle Z(\beta-it)\rangle + \langle Z(\beta+it)Z(\beta-it)\rangle_{\mathrm{c}} \,.$$

This can all be written in terms of the language of the previous section quite readily. The disconnected piece is the product of two copies of equation (108) while the connected piece is computed as follows (Banks et al, 1990):

$$\langle Z(\beta)Z(\beta')\rangle_c = \int_{-\infty}^{\mu} dx \int_{\mu}^{\infty} dy \, \langle x|e^{-\beta\mathcal{H}}|y\rangle\langle y|e^{-\beta'\mathcal{H}}|x\rangle \, , \tag{119}$$

(where you can probably see the beginnings of a pattern for how to do the correlator with an arbitrary number of insertions). Remembering $\mathcal{P}$'s definition in equation (109), we can see that there's both $\mathcal{P}$ and $\mathbf{1}-\mathcal{P}$ in play here, and:

$$\begin{aligned}\langle Z(\beta)Z(\beta')\rangle_c &= \mathrm{Tr}(e^{-\beta\mathcal{H}}(1-\mathcal{P})e^{-\beta'\mathcal{H}}\mathcal{P}) \\ &= \mathrm{Tr}(\mathcal{P}e^{-(\beta+\beta')\mathcal{H}}) - \mathrm{Tr}(e^{-\beta\mathcal{H}}\mathcal{P}e^{-\beta'\mathcal{H}}\mathcal{P}) \\ &= \langle Z(\beta+\beta')\rangle - \int dE \int dE' K(E,E')K^*(E',E)e^{-\beta E-\beta' E'} \, ,\end{aligned} \tag{120}$$

where the important object, the "Kernel" (about which I'll say more later) is:

$$K(E,E') \;=\; \int_{-\infty}^{\mu} \psi(E,x)\psi(E',x)dx \; .$$

In this setting, it is real and symmetric in its two entries, so it's really its square that appears here. The two point function becomes, on setting $\beta \to \beta + it$ and $\beta' \to \beta - it$:

$$\langle Z(\beta+it)Z(\beta-it)\rangle_c = \langle Z(2\beta)\rangle - \int dE \int dE' K(E,E')^2 e^{-\beta(E+E')-it(E-E')} \, , \tag{121}$$

which reveals a time dependent piece that in fact goes to zero at large t, leaving a time independent positive term, of magnitude $\langle Z(2\beta)\rangle$, which is the "plateau" to which the averaged quantity saturates.

For our Airy example, the spectral form factor is computable in closed form. Given the result in equation (116), we have:

$$\langle Z(\beta)\rangle\langle Z(\beta')\rangle = \frac{e^{\frac{\hbar^2}{12}(\beta^3+\beta'^3)}}{4\pi\hbar^2(\beta\beta')^{3/2}} \, , \tag{122}$$

while implementing equation (120) yields the connected piece (Okuyama, I think:

$$\langle Z(\beta)Z(\beta')\rangle_c = \frac{e^{\frac{\hbar^2}{12}(\beta+\beta')^3}}{2\pi^{1/2}\hbar(\beta+\beta')^{3/2}}\mathrm{Erf}\left(\frac{1}{2}\hbar\sqrt{\beta\beta'(\beta+\beta')}\right) \, , \tag{123}$$

and so putting $\beta \to \beta + it$ and $\beta' \to \beta - it$ yields:

$$\langle Z(\beta+it)Z(\beta-it)\rangle = \frac{e^{-\frac{1}{2}\hbar^2\beta t^2+\frac{1}{6}\hbar^2\beta^3}}{4\pi\hbar^2(\beta^2+t^2)^{\frac{3}{2}}} + \frac{e^{\frac{2\hbar^2\beta^3}{3}}}{4\sqrt{2\pi}\,\hbar\beta^{\frac{3}{2}}}\mathrm{Erf}\left(\hbar\sqrt{\frac{\beta(\beta^2+t^2)}{2}}\right) \, . \tag{124}$$

Figure 3 has a plot of the full (matrix model average) spectral form factor in black, along with a (non-averaged) sample in red.

Notice that at large t the first term dies away to zero as $t^{-3}e^{-\#t^2}$, (the power law was mentioned in Turiaci's lecture on JT gravity). Meanwhile, the last term indeed becomes $Z(2\beta)=e^{\frac{2}{3}\hbar^2\beta^3}/4\sqrt{2\pi}\hbar\beta^{3/2}$. (This last part that contains the plateau is in fact fully non-perturbative, which is hard to get in general.)

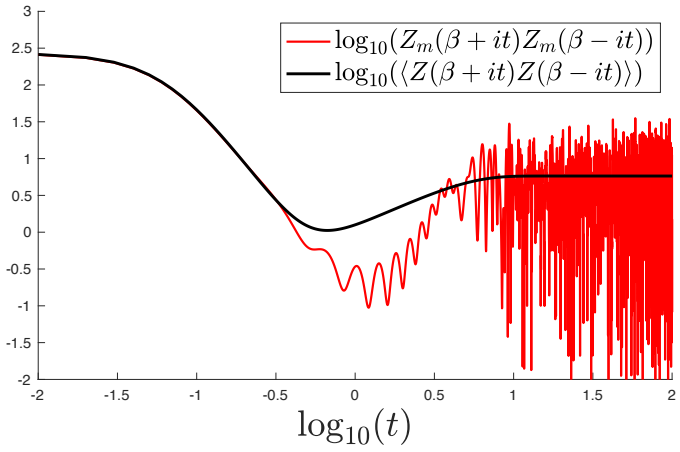

Figure 3: The spectral form factor for the mean spectrum $\{\mathcal{E}_n\}$ (red, jagged) plotted against the ensemble averaged quantity, for the Airy model (black, smooth). The two curves follow each other closed at early times before the red curve begins fluctuations. Here $\beta=\frac{1}{15}$, and $\hbar=1$.

In the current gravity parlance, the connected term is interpreted as a spacetime wormhole contribution to the gravity path integral. The early (in time t) part is the perturbative linear "ramp" behaviour that was mentioned in Turiaci's lectures as coming from computing the cylinder. This can be seen from the small s behaviour: $\mathbf{Erf}(s) \sim 2s/\sqrt{\pi} + \cdots$, giving the universal leading piece, right down to the $1/4\pi$:

$$\langle Z(\beta+it)Z(\beta-it)\rangle_{\rm c} = \frac{t}{4\pi\beta} + \cdots . \qquad \text{early time} \tag{125}$$

3.5 A wave at topological recursion

It is worth pausing to admire the perturbative expansion of the spectral density (115) in the case of Airy model:

$$\rho(E) = \frac{\sqrt{-E}}{\pi\hbar} + \frac{1}{32\pi}\frac{\hbar}{E^{5/2}} + \frac{105}{2048\pi}\frac{\hbar^3}{E^{11/2}}\cdots \tag{126}$$

The first term in expansion (126) is the contribution from one boundary (disc), while the second is the "torus" with one boundary, and so forth.

It is amusing to connect this to the topological recursion formalism, where (using the notation of Saad Shenker Stanford) writing $E=-z_1^2$ the relevant objects are the $W_{g,n}$, and appearing here will be:

$$W_{0,1} = 2z_1^2\,,\ W_{1,1} = \frac{1}{16z_1^4}\,,\ \text{and}\ W_{2,1} = \frac{105}{1024z_1^{10}}\,, \tag{127}$$

which are related to the resolvents $R_{g,1}$ defined there as $W_{g,1}=-2z_1R_{g,1}(-z_1^2)$, and so we see that

$$\pi\rho(z) = \sum_g \hbar^{2g-1}R_{g,1}(z)\,, \tag{128}$$

a toy version of what we know from JT gravity.

The $W_{g,1}$ are Laplace transforms: $W_{g,1}(z_1)=\int_0^\infty b_1 e^{-b_1z_1}V_{g,1}(b_1)$, defining the quantities that are the analogues of the Weil-Petersson quantities for the JT case:

$$V_{1,1} = \frac{b_1^2}{96}\,,\ \text{and}\ V_{2,1} = \frac{105}{1024}\frac{b_1^8}{9!}\,, \tag{129}$$

($V_{0,1}$ is undefined).

The spectral curve (derived from the leading spectral density $\rho_0(E)$) seeds the topological recursion for all higher surfaces with g arbitrary handles and n boundaries.

In that language (see courses by Eynard and others last week) the spectral curve here is $y = z$ whereas for JT it is $y = \frac{1}{4\pi}\sin(2\pi z)$.

Amusingly, in this one boundary sector, there's an alternative way of getting all these terms, and more besides. (see Johnson '20, '24). Because of the underlying Schrödinger problem, the spectral density happens to be the integral of the resolvent $\widehat{R}(x,E)$ of $\mathcal{H}$:

$$\rho(E) = \frac{1}{\pi\hbar}\mathrm{Im}\int_{-\infty}^{0}\widehat{R}(x,E)dx \ , \tag{130}$$

which in turn satisfies an ODE called the Gel'fand Dikki equation:

$$4(u(x)-E)\widehat{R}^2 - 2\hbar^2\widehat{R}\widehat{R}'' + \hbar^2(\widehat{R}')^2 = 1 \ . \tag{131}$$

So we see that $R_{g,n}(E) = -\int_{-\infty}^{0}\widehat{R}_g(x,E)dx$. Expansion (126) can be readily developed by recursively solving this equation for $\widehat{R}(x,E)$ with a given input $u(x)$, and so evidently also implies non-perturbative information about the $W_{g,1}$ and $R_{g,1}$ too.

The nice thing is that this is not just true for the Airy case of course. Once one finds $u(x)$ (perturbatively or non-perturbatively) for any matrix model in this wide class, it is a swift matter to derive the $W_{g,1}$ and $V_{g,1}$ using this route. See (Johnson '24 and also Lowenstein '24) for examples involving JT gravity, JT supergravity, and even the Virasoro minimal strings and supersymmetric versions.

One can go further with this formalism, and the central object is the Kernel $K(E,E')$ defined earlier in (121). Its diagonal is the spectral density, so a single copy of it appeared in the one-point correllator of $Z(\beta)$ and also notice how it shows up in multi-point correllators, one for each factor of $Z(\beta)$. This suggests that we should be able to read off the $W_{g,n}$ from appropriate combinations of n powers of $K(E,E')$. (I can't be the first to have noticed this, right?)

Let's have a quick look. There's another way of writing the Kernel that is very useful (using the continuum version of the "Christoffel-Darboux" relation satisfied by orthogonal polynomials):

$$K(E,E') \ = \ \frac{\psi(\mu,E)\psi'(\mu,E') - \psi'(\mu,E)\psi(\mu,E')}{E-E'} \ ,$$

where here a prime means an E derivative. We can start by writing out the leading term in the WKB expansion (112) of $\psi(E,x=0)$, which is:

$$\psi(E) \simeq \frac{1}{\sqrt{\pi\hbar}}\frac{1}{E^{\frac{1}{4}}}\cos\left(\frac{1}{\hbar}\frac{2}{3}E^{\frac{3}{2}} - \frac{\pi}{4}\right) \ , \tag{132}$$

and since it is $K(E,E')^2$ that appears in the connected two-point function (120), we see, after a bit of work (and averaging over fast oscillations), that the leading perturbative contribution is of the form:

$$\frac{1}{(2\pi)^2}\frac{E-E'}{\sqrt{EE'}(E-E')^2} \ , \tag{133}$$

and (Saad et al) have already shown (their equations (137) - (139)) how to convert this into $R_{0,2}$. This is the term that gives, after Laplace transform, the universal cylinder term we already saw emerge by taking the small argument limit of the error function in equation (123):

$$\langle Z(\beta)Z(\beta')\rangle_{\text{cylinder}} = \frac{1}{2\pi}\frac{\sqrt{\beta\beta'}}{(\beta+\beta')} \ . \tag{134}$$

3.6 Resolving some microstructure

There's an even deeper exploration one can make into non-perturbative physics using the tools we've outlined here, and that is to mine the (statistical) details of individual energy levels of the matrix model ensemble. When applied to JT and variants expecially (Johnson'21, '22) this is probably getting as close as one can in this formulation of quantum gravity to directly resolving individual microstates.

3.6.1 Some more statistical experiments

You can start by writing a few extra lines of code in the program we're all pretending you've been writing. This will allow you to "experimentally" anticipate the results to be discussed in a moment. Instead of just histogramming the energies, keep track of whether they are the first, second, third, *etc.,* scaled energy level of your matrix draw $\boldsymbol{H}$. Then do individual histograms within those categories.

My way of doing this is in the footnote.[3] The results in figure 4 shows the histograms of the first six energy levels from 100K samples of 100×100 randomly generated Hermitian matrices.

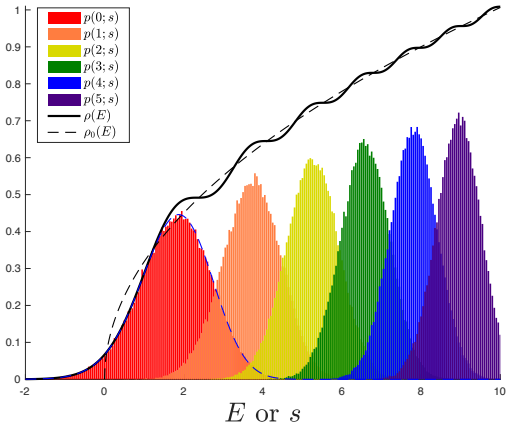

Figure 4: The spectral density $\boldsymbol{\rho(E)}$ for the Airy model (solid line). The rising dashed line is the leading result $\boldsymbol{\rho_0(E)=\sqrt{E}/(\hbar\pi)}$ for $\hbar = \mathbf{1}$. The histograms $\boldsymbol{p(n;E)}$ are frequencies of the $\boldsymbol{n}$th energy level, extracted numerically from a Gaussian random system of $\mathbf{100\times100}$ Hermitian matrices, for $\mathbf{100}\boldsymbol{K}$ samples. Note the correspondence with the undulations in $\boldsymbol{\rho(E)}$. The blue dashed peak is the exact Tracy-Widom distribution (Tracy-Widom '92) $\boldsymbol{p(0;E)}$ for the first level $\boldsymbol{E_0}$, and $\boldsymbol{\langle E_0\rangle \simeq 1.77108}$.

The point is that we've uncovered the details of the probability distributions $\boldsymbol{p(n,E)}$ for each of the individual energy levels ($\boldsymbol{n = 0,1,2,\ldots}$) in the ensemble! If you add these peaks together and just forget about which energy level ordering, and so make all the data blue, you reconstruct the spectral density plotted in figure 2.

In other words, while the spectral density is a smooth function, it also can be decomposed in a discrete way that recalls that there's underlying discrete spectrum in all of this:

$$\boldsymbol{\rho(E) = \sum_{n=0}^{\infty} p(n,E)\,.} \tag{135}$$

[3] You might still have your MATLAB window open: This can be done straightforwardly from the earlier data by adding lines of code that simply compute the ordering of all the eigenvalues of each sample (use `sort()`), and then creating separate lists that accumulate the eigenvalues according to which order they were in, per sample. Histogramming can then be done on each of these lists, and some colour coding added for good measure.

Wouldn't it be great to do this for models of gravity? We can, but we have to find the $u(x)$ that seeds everything, and that will be described in the next lecture.

3.7 Applying the doubls-scaled toolbox

But how does our toolbox be used to extract these results even for the case of the Airy model? It all comes down to our friend $K(E,E')$ again, applied to the Airy wavefunctions (105). The properties of this Airy kernel were used by Forrester '91 and Tracy-Widom '92 to characterize the probability distribution of the highest (or lowest) eigenvalue of the matrix distribution.

At the root of all this is the statistical interpretation of the random matrix model that I urged to consider above in (107). Heading back to the discrete problem, once one goes to the eigenvalues, the Dyson gas expression (52) also has the interpretation as the integral over what is (up to a constant) the "joint probability density" $\mathsf{P}_N(\lambda_1,\ldots,\lambda_N)$ for the eigenvalues to be at positions $\{\lambda_1,\ldots,\lambda_N\}$. From this one can derive the answers to questions about smaller groups of eigenvalues by simply integrating over the appropriate region (where they aren't).

So the following question can be asked in the matrix ensemble: What is the probability of finding no eigenvalue in the region $(a\le\lambda\le b)$? To work this out, as before integrate P over all λ_i, over all allowed values but the region of interest. After some work (Gaudin, '61; see the book by Mehta) it is:

$$E(0,(a,b)) = \det\left\| \delta_{nm} - \int_a^b \psi_{n-1}(\lambda)\psi_{m-1}(\lambda)d\lambda \right\|_{n,m=1}^{N} . \tag{136}$$

(Conventions: the name E here has nothing to do with energy. Also, the zero in the brackets reminds that it is the probability of "no eigenvalue".)

It was shown (Gaudin '61) that this can be wrtten in a useful alternative way. Consider the integral operator $\mathbf{K}|_{(a,b)}$ built from the kernel $K(\lambda,\kappa)$, defined as

$$K_N(\lambda_i,\lambda_j) \equiv \sum_{n=0}^{N-1} \psi_n(\lambda_i)\psi_n(\lambda_j) . \tag{137}$$

with $\psi_n(\lambda) = \mathrm{e}^{-NV(\lambda)/2}P_n(\lambda)/\sqrt{h_n}$. (This is the precursor of our friend we've already seen.)

This integral operator acts on the space $a\le\lambda\le b$ on some eigenfunctions $f(\lambda)$ according to:

$$\int_a^b K(\lambda,\kappa)f(\kappa) = \alpha f(\lambda) . \tag{138}$$

Decomposing $f(\lambda)$ in terms of the orthonormal wavefunction basis $\psi_n(\lambda)$, asome algebra shows that there are N solutions α_i defined by the characteristic equation:

$$\det\left\| \alpha\delta_{nm} - \int_a^b \psi_{n-1}(\lambda)\psi_{m-1}(\lambda)d\lambda \right\| = \prod_{n=0}^{N-1}(\alpha-\alpha_n) , \tag{139}$$

and hence the probability of interest (136) is the Fredholm determinant

$$E(0,(a,b)) = \prod_{n=0}^{N-1}(1-\alpha_n) = \det[\mathbb{I}-\mathbf{K}|_{(a,b)}] . \tag{140}$$

Let's take this techonology to the endpoint, because of the DSL.

To ask about the probability of the *first* energy of the ensemble, the interval of interest in the probability expression should be $a=-\infty$ and $b=s$, where the latter is some reference energy. The resulting determinant, and hence the probability sought, will be a function of s.

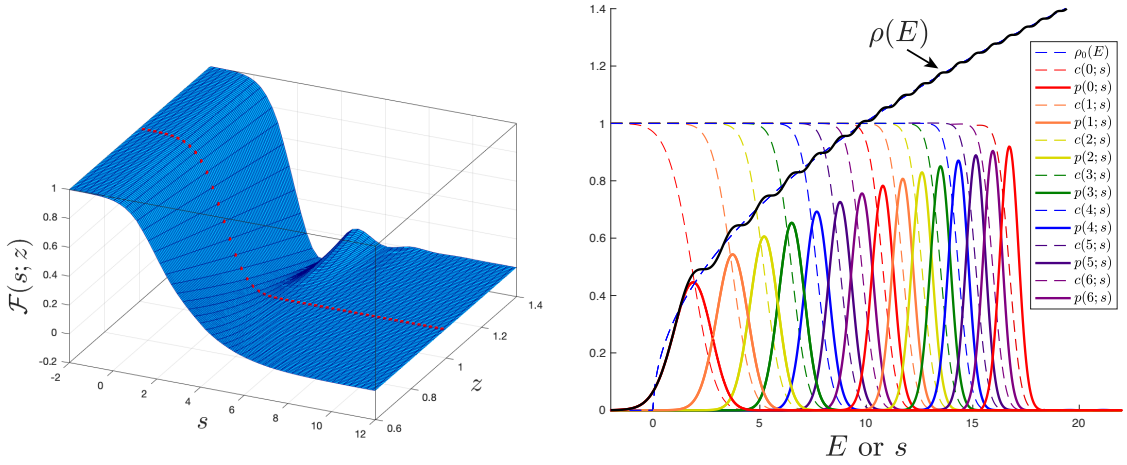

Figure 5: On the left is the Fredholm determinant $\mathcal{F}((-\infty, s); z)$, from which all of the probabilities density functions on the right for the Airy model are derived. These are the first 15 energy levels.

For the Airy prototype it is (it is usually written with a conventional subscript of 2 to denote that it is from the $\beta = 2$ Dyson-Wigner ensemble):

$$E_2(-\infty, s) = \det[\mathbb{I} - \mathrm{K_{Ai}}|_{(-\infty,s)}] \,. \tag{141}$$

The second term inside the determinant means the Fredholm integral operator deployed on the real line from $-\infty$ to s, where the Airy Kernel is used. What should be expected is as follows. Far to the left of the interval, it is unlikely that there is a eigenvalue (since everything is mostly located in the region where s (or E) is positive), and hence the result (probability there is no eigenvalue) should be close to one. As one moves more the right, the likelihood increases somewhat that some outlier might have appeared, so the "probability of none" should decrease more. Moving closer to $s = 0$ it will further decrease it since this is where the bulk of the eigenvalues starts. Moving well past $s = 0$, eigenvalues almost certainly have already appeared, and so the probability should be falling rapidly to zero. This is in fact a cumulative probability density function (CDF), and so taking a derivative with respect to s will yield the more usual probability density function (PDF) of the first eigenvalue, as will be shown below.

Even for this simple case it is hard to compute this determinant of this (infinite dimensional operator analytically). Numerics are helpful here and I learned a lot from work by Bornemann (2009) who showed that Fredholm determinants can be very efficiently computed using quadrature methods.

Before showing the results, note that a slight generalization of the above allows for the computation of the probability distribution of the second eigenvalue, the third, and so on. The notation will be as follows. Since the first eigenvalue is the ground state energy, this will be denoted the 0th energy level. The second is then the 1st level, and so forth. Little n will be the level label. Defining the object:

$$\mathcal{F}((a, b); z) = \det[\mathbb{I} - z\mathrm{K}|_{(a,b)}] \,, \tag{142}$$

it turns out that the probability for j eigenvalues to be in the interval is the result of acting with $(-d/dz)^j$ on $\mathcal{F}([a, b]; z)$, and then setting $z = 1$. The derivation of this is some straightforward (at least in the finite N case) fun with determinants involving writing $\mathcal{F}(z)$ as $\prod_i (1 - z\alpha_i)$, where $\mathcal{F}(z = 1)$ is the original Fredholm determinant $E(0, (a, b))$ of equation (140). This will be omitted here, but see Appendix A.7 of ref. [?]. From here it is plain sailing, since the probability of having the nth level appear must come from adding the probabilities of

the $(n-1)$th, $(n-2)$th, *etc.,* cases as well, (as these are independent events that must have occurred for this to be the nth level) giving that the cumulative distribution function of the nth level is:

$$c(n;(a,b)) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \frac{d^j}{dz^j} \mathcal{F}((a,b);z)\Big|_{z=1} . \tag{143}$$

and the probability distribution $p(n;(a,b))$ for the nth level can then be obtained from this by differentiation (see below).

Here is a brief summary of the methods of Bornemann mentieond earlier. The problem is to evaluate the determinant of an operator on the energy interval (a,b). Thought of as a matrix, it is infinite dimensional, and so this is an additional challenge. The technique of quadrature represents functions on an interval in terms of a basis of special functions, and reduces the problem of integrating them to a sum. This is achieved by breaking up the interval interval into m points e_i and computing weights w_i ($i = 1 \cdots m$) for each point. The values of the function at those points are weighted by the w_i and the sum gives an approximation to the integral. $\int_0^s f(E)dE \rightarrow \sum_i^m w_i f(e_i)$. The weights used depend upon the choice of special functions–each quadrature method has its own choices. The kind of quadrature that it is best to use depends upon the kinds of functions expected to be integrated on the interval. How well this works depends upon how well adapted the quadrature method is to the class of function being integrated, as well as the number of quadrature points m. This is an ancient technique that works extremely well in a wide class of cases, and underlies much of the off-the-shelf numerical methods used by computers to solve integrals. Just as the integral can be reduced to a sum, an integral operator becomes a finite matrix and so its determinant becomes a finite process that can be computed using the set of weights as used for ordinary quadrature:

$$\det(\mathrm{I} - z\mathrm{K}|_{(-\infty,s)}) \rightarrow \det(\delta_{ij} - z w_i^{\frac{1}{2}} K(e_i,e_j) w_j^{\frac{1}{2}}) . \tag{144}$$

Bornemann has shown that this works extremely well (using *e.g.,* Clenshaw-Curtis and Gauss-Legendre quadrature) for many important Fredholm problems, including problems such as the Airy model. It is an interesting exercise to reproduce such results here, in preparation for later adapting the methods for use with matrix models involving gravity. If the $\psi(E,x)$ are known analytically, as they are for Airy, it turns out that the method gives impressively accurate results for the first few levels with remarkably modest values of m such as 8, or 16, in matters of seconds.

Working on the interval $(-\infty,s)$, using the Airy Kernel, the full $\mathcal{F}$ determinant of equation (142) can be constructed, in preparation for constructing information about higher levels as well. A portion of the result is displayed in figure 5, on the left. (It was made, with m=64, on a grid of 5000 points for s and 1000 for z, and took ~81s to generate.[4]) The two-dimensional slice at z=1 (red line) shows the cumulative distribution function for the zeroth energy (ground state), $c(0;s) \equiv E(0;(-\infty,s))$, and the more general z-dependence will produce, through z-derivatives, the CDFs for higher levels, $c(n;s)$, and their PDFs $p(n;s) = -dc(n;s)/ds$. (In the notation, dependence on $(-\infty,s)$ is simplified to just dependence on s.)

See the plot on the right of figure 5 for the CDFs and PDFs for the first 15 levels. Key features are as follows:

- The rising dashed line is the perturbative result $\rho_0(E) = (\pi\hbar)^{-1} E^{\frac{1}{2}}$ obtained by scaling the Wigner semi-circle law. See equation (103).

[4] If the Reader is tempted to try this, it requires only a little bit more coding than done for this section so far. Off the shelf quadrature weights can be used, but Bornemann (2009) actually supplies some lines of code for generating the quadrature weights, and some guidance as to how to use them. It is then simply a matter of writing some instructions to generate $K(E,E')$ out of Airy functions, and implementing the quadrature on $(0,s)$.

- The solid black line is the fully non-perturbative $\rho(E)$ given in equation (115).
- The solid red line is the PDF $p(0;s)$ of the lowest eigenvalue of each matrix in the ensemble, known as the Tracy-Widom distribution. The dashed red line is the cumulative CDF $c(0;s)$ for this lowest level.
- Successive peaks in solid lines show the PDFs for the next 14 energies of the ensemble, computed using (143). Also shown are their CDFs.

So of course, it is natural to ask if all this can be done for matrix models of JT gravity and/or variants thereof and the answer is yes! (Johnson '21, '22, ...) I'll talk about it in the next lecture a little.

3.8 Looking ahead

We've explored some of the tools in a powerful toolbox together, mostly using the Airy model as a testbed. It serves as a very nice model of many of the key features of a full theory of JT gravity, or variants thereof.

Our task now is clear. Formulating JT-type gravity as a matrix model will be equivalent to finding the function $u(x)$. How do we do that? This is the subject of the last lecture.

4 Lecture 4

I will place here more detailed notes later, but you can see a topic breakdown of what I hope to cover.

4.1 JT gravity from multicritical models

4.1.1 Construction/Deconstruction

4.1.2 Meaning: A special limit

4.2 Non-perturbative issues

4.2.1 Instantons (and waving at Resurgence)

4.2.2 String Equation perspective

4.3 A way to non-perturbatively define JT gravity

4.4 Supersymmetric JT gravity from another class of multicritical models

4.4.1 $\mathcal{N}=1$ JT supergravity

Yes, much better behaved non-perturbatively than JT.

4.4.2 $\mathcal{N}=2$ and $\mathcal{N}=4$ JT supergravity

Yes, also much better behaved non-perturbatively than JT.

4.5 Closing Thoughts

I'll put several here maybe, but I might start or end with the following. Is the holographic dual of 2d gravity really fundamentally an ensemble of Hamilitonians? Or have we simply learned

that the Euclidean gravitational path integral (defined by summing over all topologies) is not the full definition of a quantum gravity?

People differ on this point. You probably have your own thoughts. For what it is worth, here's what I think at the time of writing. I think that the random matrix model appears because the GPI is a form of coarse-graining, not really the full answer. It is a Wignerian way of learning what you can about the detailed microscopic theory if you only have some parts of the Hamiltonian characterized. This is a nice way of seeing how the 't Hooftian and Wignerian ways of working with the matrix model work hand in hand. (See my essay from 2022, "Wigner meets 't Hooft at the black hole horizon" for more.)