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Les Houches lectures on non-perturbative topological strings

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Abstract: Lecture notes for the Les Houches School on Quantum Geometry

Contents

1 Introduction

In these lectures I will review some non-perturbative aspects of the topological string. There have been many takes on this problem, but my focus will be on the "resurgent" approach, and on the so-called TS/ST correspondence. Before embarking on these lectures, it is useful to have a general view of what we mean by a "non-perturbative" approach.

1.1 Perturbative and non-perturbative physics

Observables in physical theories are often functions $F(g)$ of a control parameter g or "coupling constant". There are many situations in which determining $F(g)$ for arbitrary values of g is in practice difficult. If the value of $F(g)$ is known for a reference value of g (which I will take to be $g = 0$), one can try to use perturbation methods to understand what happens when g is near this reference value, i.e. when g is small. The outcome of these methods is a perturbative series in g, of the form

$$\varphi(g) = \sum_{n\geq 0} a_n g^n. \tag{1.1}$$

(In these lectures, $\varphi(g)$ will denote a formal power series.) However, as explained by I. Aniceto in this school, more often than not the series obtained in perturbation theory are factorially divergent, i.e. the coefficients grow like $a_n \sim n!$. This means that the series $\varphi(g)$ does not define a function in a neighbourhood of $g = 0$. Rather, $\varphi(g)$ provides an asymptotic approximation to $F(g)$, in the sense of Poincaré, and we write

$$F(g) \sim \varphi(g). \tag{1.2}$$

Extracting physical information on $F(g)$ from its asymptotic expansion $\varphi(g)$ has been an important problem in physics and mathematics. In general, asymptotic series give *approximate* results, although sometimes one can find exact answers through appropriate resummation techniques.

In the above discussion we have assumed that, in our theory, $F(g)$ can be mathematically defined as an actual function, at least for some range of values of g. If this is the case, we will call $F(g)$ a *non-perturbative definition* of our observable. If $F(g)$ is well defined, the perturbative series $\varphi(g)$ can typically be obtained from this definition. However, real life turns out to be more complicated, and as we consider more and more complicated quantum theories, non-perturbative definitions become harder to obtain. Let us discuss various possible scenarios, and their realizations in physical theories.

In the best possible scenario, we have a rigorous mathematical definition of the function $F(g)$, an algorithmic procedure to calculate it for a wide range of g, and a method to obtain a perturbative expansion for small g. This is often the case in quantum mechanics. A typical example is a non-relativistic particle in the quartic potential

$$V(q) = \frac{q^2}{2} + gq^4. \tag{1.3}$$

Here, the coupling constant is g, and a typical observable is e.g. the energy of the ground state $E_0(g)$. When $g > 0$ this function is defined rigorously by the spectral theory of the self-adjoint Schrödinger operator

$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + V(\mathsf{q}), \tag{1.4}$$

where q, p are canonically conjugate Heisenberg operators. We also have numerical techniques, like Rayleigh–Ritz methods, to compute this ground state energy for arbitrary values of g. Finally, the rules of stationary perturbation theory give a power series in g for $E_0(g)$, of the form

$$\varphi(g) = \frac{1}{2} + \frac{3g}{4} - \frac{21}{8}g^2 + \cdots \tag{1.5}$$

It can be shown that this gives indeed an asymptotic expansion for $E_0(g)$. Moreover, one can use Borel resummation techniques to recover the exact $E_0(g)$ from $\varphi(g)$ (see e.g. [1] for a review and references).

In the second best scenario, one has a method to obtain perturbative expansions, and an non-rigorous algorithmic procedure to calculate $F(g)$ non-perturbatively. This is the typical situation in quantum field theory (QFT). One of the main achievements in QFT is the development of renormalized perturbation theory, which produces mathematically well-defined formal power series in a coupling constant g. For some observables we can also obtain a non-perturbative definition by using a lattice regularization of the path integral, and then taking the continuum limit. This latter procedure is in general not mathematically rigorous, but in practice seems to leads to well-defined and explicit numerical results. There is a branch of mathematical physics, called constructive QFT, whose goal is to provide mathematically rigorous non-perturbative definitions of observables, akin to what can be achieved in quantum mechanics. Recently there have been some advances in constructive QFT by using probabilistic techniques, but progress in that front has been slow and mostly in low dimensions. There are special cases in QFT in which we can obtain non-perturbative approaches by other means. For example, in integrable quantum field theories one can use the Bethe ansatz and form factor expansions to obtain non-perturbative definitions of some observables. In some theories with a large N expansion one can sometimes obtain exact results as a function of the renormalized coupling constant, albeit order by order in a series in $1/N$. An additional complication of QFTs is that the relationship between the perturbative and the non-perturbative approaches is much more complicated than in quantum mechanics, and showing that the perturbative series provides an asymptotic expansion of the available non-perturbative definitions becomes non-trivial.

The case of string theories is even more challenging, since they are defined only by perturbative expansions, and non-perturbative definitions simply do not exist in general. One can try to construct string field theories, i.e. spacetime actions whose Feynman rules reproduce ordinary perturbation theory. As reviewed in C. Johnson's lectures, there are simpler examples of string theories in low dimensions where one can use a sort of lattice regularization in terms of matrix integrals in order to define exact observables. Another class of examples concerns string theories on Anti-de Sitter backgrounds, where string theory is expected to be equivalent to a QFT. In these backgrounds, the non-perturbative definition of string theory amounts to the non-perturbative definition of the underlying QFT.

Therefore, both in QFT and in string theory we have in principle a systematic approach to compute formal power series in the coupling constant, through the rules of perturbation theory. We have a harder time in obtaining non-perturbative, exact definitions of observables. This problem becomes particularly acute in string theory. In view of this, it might be a good idea to try to extract as much information as possible from the perturbative series itself, as 't Hooft advocated [2]. It turns out that there is a framework to do this which was developed in the late 1970s by various physicists, and it was later formalized by the mathematician Jean Écalle under the name of theory of resurgence. The basic idea of the theory of resurgence is that one can obtain non-perturbative results by appropriate resummations of formal series. However, in order to do that one needs to go beyond the perturbative sector and to consider *non-perturbative effects*, which mathematically are formal power series with an additional exponentially small dependence on the coupling constant. Some of these non-perturbative effects (but in general not all) turn out to be hidden in the perturbative series, and one can learn something about the non-perturbative aspects of the theory by extracting these effects from perturbation theory.

In these lectures, our starting point will be a perturbative series $\varphi(g)$, and the search for "non-perturbative" aspects will refer to either of the following two problems:

1. *Non-perturbative effects*: given $\varphi(g)$, can we obtain an explicit description of the non-perturbative effects which are hidden in the perturbative series? More precisely, can we obtain an explicit description of the so-called *resurgent structure* of $\varphi(g)$?

2. *Non-perturbative definition*: given $\varphi(g)$, is it possible to construct a well-defined function $F(g)$ which has $\varphi(g)$ as its asymptotic expansion?

These two problems are logically independent and they have a very different flavour. The first problem has a unique solution. It is encoded in what I have called the *resurgent structure* associated to a perturbative series. However, obtaining explicit descriptions of resurgent structures turns out to be a very difficult problem, even in simple quantum theories. Given only the perturbative series, the second problem clearly does not have a unique solution, since there are infinitely many functions with the same asymptotic expansion. Therefore, the relevant question is whether there is a way to pick up a particular non-perturbative definition as the one which is physically relevant, or perhaps the one which is more interesting mathematically.

Although the two problems above can be addressed separately, they are also related, in the sense that once a solution to the second problem has been found and a non-perturbative definition is available, one can ask whether it can be reconstructed by using the resurgent structure, i.e. by using the non-perturbative effects obtained in the solution to the first problem.

1.2 Non-perturbative topological strings

Topological string theory can be regarded as a simplified model of string theory, more complex than non-critical string theories, but still simpler than full-fledged string theories. Topological string theories are interesting for various reasons. They provide a physical counterpart to the theory of enumerative invariants for Calabi–Yau (CY) threefolds, and they have led to many surprising results in that field. Through the idea of geometric engineering [3], they are closely related to $\mathcal{N} = 2$ supersymmetric gauge theories in four and five dimensions. They have multiple connections to classical and quantum integrable models, as seen in e.g. [4–7]. Finally, they lead to simpler but precise realizations of large N dualities, in which the large N dual can be a matrix model [8, 9], a Chern–Simons gauge theory [10], or, as we will see in these lectures, a one-dimensional quantum-mechanical model [11, 12]. For these reasons, there have been various efforts to understand non-perturbative aspects of topological strings from different perspectives. In these lectures we will address this problem by first considering the resurgent structure of topological strings, as proposed in question 1 above, and then addressing the question 2 of finding an interesting non-perturbative definition.

The structure of these lectures is the following. I will first review some properties of perturbative topological strings in section 2. This section is far from self-contained, and a detailed derivation of all the results I will mention would be the topic of various summer schools. It should be regarded as a list of properties which will be useful later on. In section 3 we address their resurgent structure, and we will essentially answer the question 1 above. In section 4, we provide a possible answer to question 2 above in the case of toric CY threefolds, namely, we consider a well-defined function, obtained from a quantum-mechanical problem, whose asymptotic expansion conjecturally reproduces the perturbative series of the topological string. Appendix A summarizes some useful results in the theory of resurgence which I will use in the lectures. Appendix B list some useful properties of Faddeev's quantum dilogarithm.

2 Perturbative topological strings

In this section I give some background on perturbative topological strings. This is a big subject which would require many lectures in itself, so I cannot cover all the details. Useful references about topological string theory include [13–16]. A very incomplete but hopefully useful set of lectures can be found in [17]. Various aspects of the mathematical view on topological strings have been presented in the lectures of M. Liu in this school.

Topological string theory, as we mentioned above, can be regarded as a toy model of string theory. Since a string theory is a quantum theory of maps from Riemann surfaces to a target manifold, we have so specify first our target. This will be a CY threefold, i.e. a complex, Kähler,

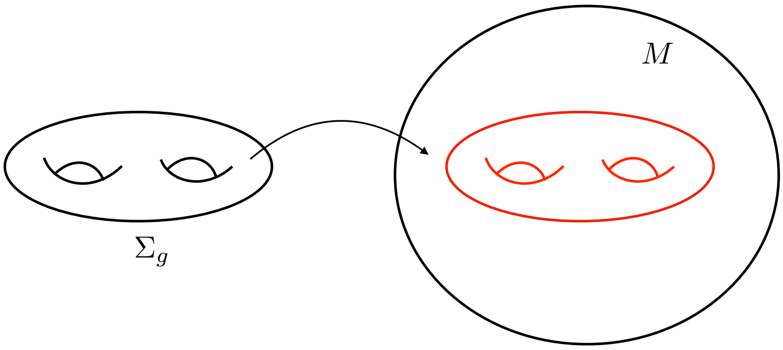

Figure 1. A pictorial representation of a holomorphic map from a Riemann surface Σ_g into a CY M.

Ricci-flat manifold of complex dimension three (other choices are possible, but have been studied less intensively). We will denote such a target by M, and we emphasize that M does not have to be compact. In fact, non-compact CY threefolds will be very important for us, for various reasons.

The starting point to construct topological string theory is the $\mathcal{N} = 2$ supersymmetric version of the non-linear sigma model, with target space M. This model can be topologically twisted to obtain a topological field theory in two dimensions [18]. Topological string theory is then obtained by coupling the resulting twisted theory to topological gravity, in the way explained in [19, 20]. There are however two different ways of twisting the non-linear sigma model, known as the A and the B twist [21, 22], and this leads to two different versions of topological string theory, which we will call the A and the B model. These models are sensitive to different properties of M. The A model is sensitive to the Kähler parameters of M, which specify the (complexified) sizes of the two-cycles in M. There are $h^{1,1}(M) = b_2(M)$ Kähler parameters in total, where $h^{p,q}(M)$ are the Hodge numbers of M, and $b_i(M)$ are its Betti numbes. We will denote these parameters by t_i, $i = 1, \cdots, s$, where $s = h^{1,1}(M)$, and we will gather them in a vector $\mathbf{t} = (t_1, \cdots, t_s)$. The B model is sensitive to the complex parameters of M, which specify its "shape", and there are

$$h^{1,2}(M) = \frac{b_3(M)}{2} - 1 \tag{2.1}$$

complex moduli in total. We will denote them by z_i, $i = 1, \cdots, h^{1,2}(M)$.

Mirror symmetry is a duality or equivalence between the A and the B models, which we will assume in the following. This means in particular that there is a map between the Kähler and the complex moduli of the mirror manifolds, which we can write as $t_i = t_i(z_1, \cdots, z_s)$, $i = 1, \cdots, s$. This map is usually called the *mirror map*, and we will see explicit examples below.

The only observable of topological string theory on a CY threefold is the partition function, or its logarithm the free energy. The latter can be calculated as a perturbative series by summing over connected Riemann surfaces. The contribution of a genus g Riemann surfaces to the free energy will be denoted by F_g, and it is a function of the Kähler (respectively, complex) moduli in the A (respectively, B) model.

2.1 The A model

To solve topological string theory perturbatively, one has to calculate F_g for all $g \geq 0$. Let us describe how to do this calculation in the A model. Since the theory is topological, one can show that it just “counts” instantons of the twisted non-linear sigma model with target M. The instantons are in this case holomorphic maps from the Riemann surface of genus g to the CY M,

$$f : \Sigma_g \to M, \tag{2.2}$$

see Fig. 1 for a pictorial representation. Let $[S_i] \in H_2(M, \mathbb{Z})$, $i = 1, \cdots, s$, be a basis for the two-homology of M, with $s = b_2(M)$ as before. The maps (2.2) are classified topologically by the homology class

$$f_*[(\Sigma_g)] = \sum_{i=1}^{s} d_i [S_i] \in H_2(X, \mathbb{Z}), \tag{2.3}$$

where d_i are integers called the *degrees* of the map. We will put them together in a degree vector $\mathbf{d} = (d_1, \cdots, d_s)$. The “counting” of instantons is given by the *Gromov–Witten (GW) invariant* at genus g and degree $\mathbf{d}$, which we will denote by $N_g^{\mathbf{d}}$, and is given by an appropriate integral over the space of collective coordinates of the instanton, or moduli space of maps, as explained in M. Liu’s lectures (see also [14] for definitions and examples). Note that GW invariants are in general rational, rather than integer, numbers.

The genus g free energies can be essentially computed, as an expansion near the so-called *large radius point* $\mathbf{t} \to \infty$ if you know all the GW $F_g(\mathbf{t})$ at genus g and at all degrees. They are given by formal power series in e^{-t_i}, $i = 1, \cdots, s$, where t_i are the Kähler parameters of M. They also involve additional contributions, which are polynomials in the t_i. At genus zero, the free energy reads

$$F_0(\mathbf{t}) = \frac{1}{6} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{\mathbf{d}} N_{0,\mathbf{d}} \mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}. \tag{2.4}$$

In the case of a compact CY threefold, the numbers a_{ijk} are interpreted as triple intersection numbers of two-classes in X, and one usually adds an additional polynomial of degree two in the Kähler parameters, but we will not busy ourselves with these details here. At genus one, one has

$$F_1(\mathbf{t}) = \sum_{i=1}^{s} b_i t_i + \sum_{\mathbf{d}} N_{1,\mathbf{d}} \mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}. \tag{2.5}$$

In the compact case, the coefficients b_i are related to the second Chern class of the CY manifold [23]. At higher genus one finds

$$F_g(\mathbf{t}) = c_g \chi + \sum_{\mathbf{d}} N_{g,\mathbf{d}} \mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}, \qquad g \geq 2. \tag{2.6}$$

Here, χ is the Euler characteristic of M in the compact case, and a suitable generalization thereof in the non-compact case. The constant term $c_g \chi$ in (2.6) is the contribution of *constant maps* to the genus g free energy. The coefficient c_g is given by an integral over the moduli space of Riemann surfaces [20], whose value can be obtained by using string dualities [24] or by a direct calculation [25],

$$c_g = \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{4g(2g-2)(2g-2)!}. \tag{2.7}$$

Although the genus g free energies have been written in (2.4), (2.5), (2.6) as formal power series, they have a common region of convergence near the large radius point $t_i \to \infty$, and therefore they define actual functions $F_g(\mathbf{t})$, at least near that point in moduli space. The total free energy of the topological string is formally defined as the sum,

$$F(\mathbf{t}; g_s) = \sum_{g \geq 0} g_s^{2g-2} F_g(\mathbf{t}). \tag{2.8}$$

This can be further decomposed as

$$F(\mathbf{t}; g_s) = F^{(\mathrm{p})}(\mathbf{t}; g_s) + \sum_{g \geq 0} \sum_{\mathbf{d}} N_{g,\mathbf{d}} \mathrm{e}^{-\mathbf{d} \cdot \mathbf{t}} g_s^{2g-2}, \tag{2.9}$$

where

$$F^{(\mathrm{p})}(\mathbf{t}; g_s) = \frac{1}{6g_s^2} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{i=1}^{s} b_i t_i + \chi \sum_{g \geq 2} c_g g_s^{2g-2}. \tag{2.10}$$

is the polynomial part of the free energies. The variable g_s, called the *topological string coupling constant*, is in principle a formal variable, keeping track of the genus of the Riemann surface. However, in string theory this constant has a physical meaning, and measures the strength of the string interaction: when g_s is very small, the contribution to the free energy is dominated by Riemann surfaces of low genus; as g_s becomes large, the contribution of higher genus Riemann surfaces becomes important.

If we fix the value of $\mathbf{t}$ inside the common radius of convergence of the free energies, the sum over genera in (2.8) defines a formal power series in the string coupling constant, whose coefficients $F_g(\mathbf{t})$ are functions of the moduli. Understanding the detailed properties of this series will be one of the central goals of these lectures. There is strong evidence that the series $F_g(\mathbf{t})$, at a fixed value of $\mathbf{t}$, diverges doubly-factorially,

$$F_g(\mathbf{t}) \sim (2g)!, \tag{2.11}$$

therefore the total free energy (2.8) does *not* define a function of g_s and $\mathbf{t}$.

2.2 The Gopakumar–Vafa representation

The expression in the r.h.s. of (2.9) is a double expansion, in both degrees and genera. To obtain (2.8) we sum first over all the degrees at fixed genus, to obtain the functions $F_g(\mathbf{t})$, and then we sum over genera. As we will see, this is the natural answer that one obtains from mirror symmetry and the B-model. But perhaps one could try to exchange the order of summations, i.e. to sum over all genera for a fixed $\mathbf{d}$. This was done by Gopakumar and Vafa in [26], by thinking about the physical content of the free energies in M-theory. Let us consider the double expansion in (2.9) involving the GW invariants,

$$\sum_{g \geq 0} \sum_{\mathbf{d}} N_{g,\mathbf{d}} \, \mathrm{e}^{-\mathbf{d} \cdot \mathbf{t}} g_s^{2g-2}. \tag{2.12}$$

Then, [26] found that this series can be re-expressed as

$$F^{\mathrm{GV}}(\mathbf{t}; g_s) = \sum_{g \geq 0} \sum_{\mathbf{d}} \sum_{w=1}^{\infty} \frac{1}{w} n_g^{\mathbf{d}} \left(2 \sin \frac{w g_s}{2} \right)^{2g-2} \mathrm{e}^{-w \mathbf{d} \cdot \mathbf{t}}, \tag{2.13}$$

where $n_g^{\mathbf{d}}$ are the so-called *Gopakumar–Vafa (GV) invariants*. In contrast to the GW invariants, they turn out to be *integer* numbers. They can be interpreted, roughly, as Euler characteristics of moduli spaces of D2 branes in the CY manifold. One important property of the GV invariants is that, for a given degree $\mathbf{d}$, there is a maximal genus $g_{\max}(\mathbf{d})$ such that $n_g^{\mathbf{d}} = 0$ for $g > g_{\max}(\mathbf{d})$.

Exercise 2.1. Show that the expression (2.13) leads to the following formula for $F_g(\mathbf{t})$ [27]. For $g = 0$, $g = 1$, one has

$$\begin{aligned} F_0(\mathbf{t}) &= F_0^{(\mathrm{p})}(\mathbf{t}) + \sum_{\mathbf{d}} n_0^{\mathbf{d}} \, \mathrm{Li}_3\left(\mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}\right), \\ F_1(\mathbf{t}) &= F_1^{(\mathrm{p})}(\mathbf{t}) + \sum_{\mathbf{d}} \left(\frac{n_0^{\mathbf{d}}}{12} + n_1^{\mathbf{d}}\right) \mathrm{Li}_1\left(\mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}\right), \end{aligned} \tag{2.14}$$

while for $g \geq 2$ one has

$$\begin{aligned} F_g(\mathbf{t}) &= F_g^{(\mathrm{p})}(\mathbf{t}) \\ &+ \sum_{\mathbf{d}} \left(\frac{(-1)^{g-1} B_{2g} n_0^{\mathbf{d}}}{2g(2g-2)!} + \frac{2(-1)^g n_2^{\mathbf{d}}}{(2g-2)!} + \cdots - \frac{g-2}{12} n_{g-1}^{\mathbf{d}} + n_g^{\mathbf{d}} \right) \mathrm{Li}_{3-2g}\left(\mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}\right). \end{aligned} \tag{2.15}$$

In these expressions,

$$\mathrm{Li}_n(z) = \sum_{k\geq 1} \frac{z^k}{k^n} \tag{2.16}$$

is the polylogarithm function of order n. □

It follows from (2.15) that if one knows the GW invariants $N_{g',\mathbf{d}'}$ with $g' \leq g$, $\mathbf{d}' \leq \mathbf{d}$, one can determine uniquely the GV invariant $n_g^{\mathbf{d}}$, and viceversa. In that sense, the two sets of invariants contain the same information. Let us note that there exist direct mathematical constructions of the GV invariants as well, see e.g. [28].

The GV representation of the total free energy gives another view on the problem of resummation. It is clear that we can now write the non-trivial part of the total free energy, involving the GW invariants, as

$$F^{\mathrm{GV}}(\mathbf{t}; g_s) = \sum_{\mathbf{m}} F_{\mathbf{m}}(g_s) \mathrm{e}^{-\mathbf{m}\cdot\mathbf{t}}, \tag{2.17}$$

where

$$F_{\mathbf{m}}(g_s) = \sum_{g\geq 0} \sum_{\mathbf{m}=w\mathbf{d}} \frac{1}{w} n_g^{\mathbf{d}} \left(2\sin\frac{w g_s}{2}\right)^{2g-2}. \tag{2.18}$$

Note that, due to the vanishing property of the GV invariants mentioned above, the sum over g is finite. One could think that (2.17) can perhaps be summed, as a series now in the variables e^{-t_i}. It turns out that the properties of this series depend crucially on the value of g_s. If g_s is *real*, the series is not even well-defined. This is due to the inverse square sines appearing in (2.13), which lead to singularities at rational values of g_s. In fact, given any rational value of g_s, there is a minimum degree $\mathbf{m}_{\min}$ such that infinitely many coefficients $F_{\mathbf{m}}(g_s)$ with $\mathbf{m} > \mathbf{m}_{\min}$ are singular at that rational value. As a consequence, given any real value of g_s, rational or not, there is a degree starting from which infinitely many coefficients $F_{\mathbf{m}}(g_s)$ can be made arbitrarily large. This is clearly a pathological situation. One could still try to make sense of the r.h.s. of (2.17) for complex values of g_s. However, in that case there is evidence that the coefficients $F_{\mathbf{m}}(g_s)$ grow with Gargantuan speed, even worst than factorial. For the CY manifold known as local $\mathbb{P}^2$, where $b_2(M) = 1$, and therefore there is a single degree, explicit computations suggest that

$$\log|F_m(g_s)| \sim m^2, \qquad m \gg 1. \tag{2.19}$$

Therefore, when g_s is real the GV series does not make sense, and when g_s is complex the series makes sense, but it diverges wildly. This underappreciated fact shows that the GV representation of the free energy does not define topological strings non-perturbatively.

Exercise 2.2. M. Liu's lectures have introduced the topological vertex of [29], which can be used to compute the coefficients $a_m(g_s)$ efficiently in the case of local $\mathbb{P}^2$. One finds, for the very first values of $m = 1, 2, 3$,

$$
\begin{aligned}
F_1(g_s) &= -\frac{3q^2}{(q^2-1)^2}, \\
F_2(g_s) &= \frac{3q^2\left(4q^4+7q^2+4\right)}{2\left(q^4-1\right)^2}, \\
F_3(g_s) &= -\frac{10q^{12}+27q^{10}+54q^8+62q^6+54q^4+27q^2+10}{(q^6-1)^2},
\end{aligned}
\tag{2.20}
$$

where $q = \mathrm{e}^{\mathrm{i}g_s/2}$. Write a computer program which calculates these coefficients, and extract from them the very first GV invariants of this geometry. Verify the asymptotic behaviour (2.19). □

Remark 2.3. There is a special class of toric CY manifolds which can be used to engineer five-dimensional $SU(N)$ gauge theories [3, 30]. For these manifolds, the gauge theory instanton partition function (reviewed in N. Nekrasov's lectures in this school) provides a rearrangement of the GV expansion which converges for complex g_s [31]. However, it is still singular if $g_s \in 2\pi\mathbb{Q}$.

2.3 An example: the resolved conifold

Before going on, let us consider what is perhaps the simplest example of a topological string theory, on the non-compact CY manifold known as the *resolved conifold*. This manifold is a plane bundle over the two-sphere:

$$
X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1. \tag{2.21}
$$

There is a single modulus t, which in the A-model is the (complexified) area of the $\mathbb{P}^1$. A direct calculation of the Gromov–Witten invariants in [25] shows that there is a single non-zero GV invariant in this geometry, with $g = 0$ and $d = 1$, and equal to 1. The all-genus free energy of the topological string in this case is simply given by

$$
F^{\mathrm{GV}}(t; g_s) = \sum_{w=1}^{\infty} \frac{1}{w}\left(2\sin\frac{wg_s}{2}\right)^{2g-2} \mathrm{e}^{-wt}. \tag{2.22}
$$

The full free energy differs from this expression in a cubic polynomial in t which is not important right now. Up to such a polynomial, one finds for the free energies at fixed genus,

$$
\begin{aligned}
F_0(t) &= \mathrm{Li}_3(\mathrm{e}^{-t}), \\
F_1(t) &= \frac{1}{12}\mathrm{Li}_1(\mathrm{e}^{-t}), \\
F_g(t) &= \frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}\mathrm{Li}_{3-2g}(\mathrm{e}^{-t}), \qquad g \geq 2.
\end{aligned}
\tag{2.23}
$$

These can be obtained from the general expression (2.15) by taking into account that there is a single non-vanishing GV invariant $n_0^d = 1$. It is easy to see that this series diverges doubly-factorially, by using the formula

$$\mathrm{Li}_{3-2g}(\mathrm{e}^{-t}) = \Gamma(2g-2) \sum_{k \in \mathbb{Z}} \frac{1}{(2\pi k \mathrm{i} + t)^{2g-2}}, \tag{2.24}$$

which is valid for $\mathrm{Re}(s) > 0$ and $\mathrm{e}^{-t} \neq 1$.

An even simpler topological string theory can be obtained when we look at the limit of $F_g(t)$ as $t \to 0$ and we keep the most singular terms. One finds,

$$\begin{aligned} F_0(\lambda) &= \frac{\lambda^2}{2}\left(\log(\lambda) - \frac{3}{2}\right) + \mathcal{O}(1), \\ F_1(\lambda) &= -\frac{1}{12}\log(\lambda) + \mathcal{O}(1), \\ F_g(\lambda) &= \frac{B_{2g}}{2g(2g-2)}\lambda^{2-2g} + \mathcal{O}(1), \quad g \geq 2. \end{aligned} \tag{2.25}$$

where $\lambda = \mathrm{i}t$. The point $t = 0$ (or $\lambda = 0$) is a special point in the moduli space of the resolved conifold, in which the area of the $\mathbb{P}^1$ shrinks to zero size, and the theory is singular. Such a point in moduli space is called a *conifold point*. Conifold points arise generically in the moduli space of CY manifolds, and they will play an important role in what follows.

2.4 The B model

In the B model, the problem of calculating the free energies is very different, since the twisted sigma model localizes to constant maps [21], so the calculation is in a sense "classical". In the case of genus zero, the problem is completely solved by calculating the periods of the holomorphic 3-form Ω on the CY M, as explained in the pioneering paper [32]. One chooses a symplectic basis of three-cycles,

$$A^I, \qquad B_I, \qquad I = 0, 1, \cdots, s, \tag{2.26}$$

which satisfy

$$\begin{aligned} \langle A^I, A^J \rangle &= \langle B^I, B^J \rangle = 0, \\ \langle A^I, B_J \rangle &= -\langle B_I, A^J \rangle = \delta^I_J, \end{aligned} \tag{2.27}$$

where $I, J = 0, 1, \cdots, s$ and $\langle \cdot, \cdot \rangle$ is the intersection pairing in $H_3(M)$. Integration of Ω over these cycles gives the A and B periods,

$$X^I = \int_{A^I} \Omega, \qquad \mathcal{F}_I = \int_{B_I} \Omega. \tag{2.28}$$

From here one defines a projective prepotential $\mathfrak{F}$ by the relations

$$\mathcal{F}_I = \frac{\partial \mathfrak{F}}{\partial X^I}. \tag{2.29}$$

We can now construct the so-called *flat coordinates* t^a as affine coordinates corresponding to the projective coordinates X^I: we choose a nonzero period, say X^0, and we consider the quotients

$$t^a = \frac{X^a}{X^0}, \qquad a = 1, \cdots, s. \tag{2.30}$$

Since the projective prepotential is homogeneous, we can define a quantity $F_0(\mathbf{t})$ (called the *prepotential*) which only depends on the coordinates t^a

$$\mathfrak{F}(X^I) = (X^0)^2 F_0(t\mathbf{t}). \tag{2.31}$$

The prepotential gives the genus zero free energy of the topological string, in the B model. An important bonus of using the B model is that the genus zero free energy is regarded as a *global function* on the moduli space of the CY manifold.

One basic result of mirror symmetry is that, for an appropriate choice of the symplectic basis of three-cycles, the flat coordinates give the mirror map, i.e. the t^a can be regarded as complexified Kähler coordinates of the mirror manifold to M, which we will denote by $M^\star$. For that choice, the prepotential defined above agrees with the genus zero free energy of the A model on $M^\star$. In particular, the expansion of that prepotential around the large radius point leads to the GW invariants of the mirror CY.

In the case of the so-called local CY manifolds, which are mirror to toric CY manifolds, the above construction becomes simpler. In the local case, the equation for the mirror CY is of the form

$$uv = P(\mathrm{e}^x, \mathrm{e}^y), \tag{2.32}$$

where $P(\mathrm{e}^x, \mathrm{e}^y)$ is a polynomial in the exponentiated variables x, y. There is a precise algorithm to obtain this polynomial, starting with the description of a toric CY manifold as a symplectic quotient, see e.g [33, 34]. The geometry of the threefold (2.32) is encoded in the Riemann surface Σ described by P,

$$P(\mathrm{e}^x, \mathrm{e}^y) = 0. \tag{2.33}$$

It can be shown that, in the local case (2.32), the periods of Ω reduce to the periods of the differential

$$\lambda = y(x)\mathrm{d}x \tag{2.34}$$

on the curve (2.33) [3, 33]. In addition, one can set $X^0 = 1$. The flat coordinates t^a and the genus zero free energy $F_0(\mathbf{t})$ in the large radius frame are determined by choosing an appropriate symplectic basis of one-cycles on the curve, α^a, β_a, $a = 1, \cdots, g_\Sigma$, and one finds

$$t^a = \oint_{\alpha^a} \lambda, \qquad \frac{\partial F_0}{\partial t^a} = \oint_{\beta_a} \lambda, \qquad i = 1, \cdots, g_\Sigma, \tag{2.35}$$

where g_Σ is the genus of the mirror curve. We note that, in general, $s \geq g_\Sigma$, and there are additional $s - g_\Sigma$ moduli of the CY that are obtained by considering in addition residues of poles at infinity (these additional parameters are sometimes called *mass parameters*).

Another important consequence of using the B model is that there is in fact an infinite family of flat coordinates and genus zero free energies, depending on the choice of a basis of three cycles. Different choices are related by symplectic transformations. This structure is present in the general, compact case, but in order to make things simpler, I will focus on the local case and in addition I will assume that $g_\Sigma = 1$. Then, a symplectic transformation of the cycles induces the following transformation of the periods,

$$\begin{pmatrix} \partial_t F_0 \\ t \end{pmatrix} \to \begin{pmatrix} \partial_{\tilde{t}} \widetilde{F}_0 \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \partial_t F_0 \\ t \end{pmatrix} + \begin{pmatrix} b \\ a \end{pmatrix}, \tag{2.36}$$

where

$$\alpha\delta - \beta\gamma = 1. \tag{2.37}$$

This transformation is a combination of an $\mathrm{SL}(2,\mathbb{R})$ transformation and a shift. The shift is due to the fact that there is a constant period, independent of the moduli, which can mix with the non-trivial periods. The genus zero free energy transforms through a generalized Legendre transform,

$$\widetilde{F}_0(\tilde{t}) = F_0(t) - S(t,\tilde{t}). \tag{2.38}$$

The function $S(t,\tilde{t})$ has the form

$$S(t,\tilde{t}) = \lambda t^2 + \mu t\tilde{t} + \nu \tilde{t}^2 + \hat{a}t + \hat{b}\tilde{t}. \tag{2.39}$$

The coefficients appearing in this polynomial in terms of the parameters appearing in (2.36) by imposing that $\widetilde{F}_0(\tilde{t})$ is independent of t,

$$\frac{\partial F_0}{\partial t} = \frac{\partial S}{\partial t} = 2\lambda t + \mu\tilde{t} + \hat{a}. \tag{2.40}$$

and by using that

$$\frac{\partial \widetilde{F}_0}{\partial \tilde{t}} = -\frac{\partial S}{\partial \tilde{t}}. \tag{2.41}$$

By comparing these two equations to the equations for $\tilde{t}$ and $\partial_{\tilde{t}}\widetilde{F}_0$ from (2.36), one eventually obtains

$$S(t,\tilde{t}) = -\frac{\delta}{2\gamma}t^2 + \frac{1}{\gamma}t\tilde{t} - \frac{\alpha}{2\gamma}\tilde{t}^2 - \frac{a}{\gamma}t + \left(\frac{\alpha}{\gamma}a - b\right)\tilde{t}. \tag{2.42}$$

In this derivation we have assumed that $\gamma \neq 0$. The different choices of genus zero free energy (or of flat coordinate $\tilde{t}$ in (2.35)) are usually called choices of *frame*. They are all related by this type of transformations, and they contain the same information.

The local case has additional advantages, since due to the "remodeling conjecture" [9, 35] (reviewed in the lectures by M. Liu), the higher genus free energies can be obtained through the topological recursion of Eynard–Orantin [36]. As a consequence of this, it can be shown that, under a symplectic transformation, the total free energy changes by a generalized Fourier transform (this was first postulated in [37]). Let us write down the result in the simple case with $g_\Sigma = 1$. One has

$$\exp(\widetilde{F}(\tilde{t};g_s) = \int \exp\left(F(t;g_s) - \frac{1}{g_s^2}S(t,\tilde{t})\right)\mathrm{d}t, \tag{2.43}$$

where the function $S(t,\tilde{t})$ implementing the transform is given by (2.42). The integral appearing here has to be understood formally, since the total free energy appearing in the exponent in the integrand is itself a formal power series. To obtain the transformation properties of the genus g free energies, we evaluate the integral in the r.h.s. of (2.43) in a saddle point approximation for g_s small. At leading order we recover the generalized Legendre transform (2.38), and in particular the condition for a saddle point is precisely (2.40). The evaluation at higher orders leads to explicit transformation properties for the higher genus free energies.

As we mentioned above, in the moduli space of CY manifolds there are generically conifold loci, which are characterized by the shrinking of a three-cycle with the topology of a three-sphere $\mathbb{S}^3$, and leading to a vanishing period (in the local case we have a vanishing one-cycle in the mirror curve). Like before, we will restrict for simplicity to CYs with a one-dimensional moduli space, where a conifold locus is in fact a conifold point. There is a particularly important frame in topological string theory defined by the property that the local coordinate is a vanishing period at the conifold point. This frame is called the *conifold frame*. It turns out that the genus g

free energies in that frame have the universal behaviour (2.25) near the conifold point, for an appropriate normalization of the vanishing period λ. This is an important property of topological strings, first noted in [38], where a physical explanation thereof was also proposed. In the case of the resolved conifold, the conifold coordinate λ coincides (up to a factor of i) with the large radius coordinate t measuring the size of the $\mathbb{P}^1$ in the geometry, but in general they are different, and related by a non-trivial symplectic transformation. We will see an example in the next section.

2.5 A more complicated example: local $\mathbb{P}^2$

In order to better understand the B model approach to topological string theory, it is useful to look at a rich example. An all-times favourite is the mirror to the local $\mathbb{P}^2$ CY manifold. Useful information on local $\mathbb{P}^2$ can be found in many papers, like e.g. [33, 39]. This toric CY is given by the total space of the canonical bundle of $\mathbb{P}^2$,

$$X = \mathcal{O}(-3) \to \mathbb{P}^2, \tag{2.44}$$

and appeared in M. Liu's lectures. The corresponding mirror curve is given by

$$\mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-x-y} + \kappa = 0. \tag{2.45}$$

Here, κ is a complex variable parametrizing the complex structure of the curve. It is also useful to introduce

$$z = \kappa^{-3}. \tag{2.46}$$

The curve (2.45) is in fact an elliptic curve in exponentiated variables.

Exercise 2.4. Use the transformation

$$\begin{aligned} x &= -\frac{\kappa}{2} + \frac{bY - a/2}{X + c}, \\ y &= \frac{a}{X + c}. \end{aligned} \tag{2.47}$$

to put the curve (2.45) in Weierstrass form,

$$Y^2 = 4X^3 - g_2 X - g_3. \tag{2.48}$$

Calculate g_2 and g_3, and verify that the discriminant of the curve is given by

$$\Delta(\kappa) = \frac{1 + 27\kappa^3}{\kappa}. \tag{2.49}$$

□

The exercise above shows that there are three special points in the curve (2.45). The first one is $z = 0$, or $\kappa = \infty$. As we will see in a moment, this is the large radius point of the geometry, when the complexified Kähler parameter is large. The second special point is $z = \infty$, or $\kappa = 0$. This is the so-called *orbifold point*, where the theory can be described as a perturbed topological CFT. Finally, we have the point

$$z = -\frac{1}{27}, \tag{2.50}$$

where the discriminant (2.49) vanishes and the curve is singular. This is a conifold point, similar to the point $t = 0$ in the resolved conifold. The moduli space of local $\mathbb{P}^2$, with these three special

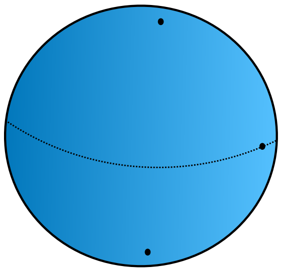

Figure 2. The moduli space of local $\mathbb{P}^2$, with the three special points $z = 0$ (large radius), $z = -1/27$ (conifold) and $z = \infty$ (orbifold).

points, is represented in Fig. 2. The upper hemisphere including the large radius point can be regarded as the "geometric" phase of the model, where the topological string can be represented in terms of embedded Riemann surfaces. In the lower hemisphere, around the orbifold point, the Gromov–Witten expansion around $z = 0$ does no longer converge, and one should use a more abstract picture in terms of a perturbed topological CFT coupled to gravity. See [40] for a discussion of the physics and mathematics of these moduli spaces.

The periods (2.35) are functions of z. The most convenient way to calculate these periods, as is well-known in mirror symmetry, is to find an ODE, or Picard–Fuchs (PF) equation, satisfied by all the periods. In the case of local $\mathbb{P}^2$, the PF equation reads [33]

$$\left(\theta^3 - 3z(3\theta+2)(3\theta+1)\theta\right)\Pi = 0, \tag{2.51}$$

where

$$\theta = z\frac{\mathrm{d}}{\mathrm{d}z} \tag{2.52}$$

and Π is a period. This equation has three independent solutions at $z = 0$, which can be calculated explicitly with the Frobenius method: a trivial, constant solution, a logarithmic solution $\varpi_1(z)$, and a double logarithmic solution $\varpi_2(z)$. If we introduce the power series,

$$\begin{aligned} \widetilde{\varpi}_1(z) &= \sum_{j\geq 1} 3\frac{(3j-1)!}{(j!)^3}(-z)^j, \\ \widetilde{\varpi}_2(z) &= \sum_{j\geq 1} \frac{18}{j!}\frac{\Gamma(3j)}{\Gamma(1+j)^2}\left\{\psi(3j)-\psi(j+1)\right\}(-z)^j, \end{aligned} \tag{2.53}$$

where $\psi(z)$ is the digamma function, we have that

$$\begin{aligned} \varpi_1(z) &= \log(z) + \widetilde{\varpi}_1(z), \\ \varpi_2(z) &= \log^2(z) + 2\widetilde{\varpi}_1(z)\log(z) + \widetilde{\varpi}_2(z). \end{aligned} \tag{2.54}$$

It is easy to see that the series in (2.53) have a radius of convergence $|z| = 1/27$, determined by the position of the conifold point. One can write $\widetilde{\varpi}_1(z)$ as a generalized hypergeometric function,

$$\widetilde{\varpi}_1(z) = -6z\, {}_4F_3\left(1,1,\frac{4}{3},\frac{5}{3};2,2,2;-27z\right). \tag{2.55}$$

We can now ask which combinations of the periods above lead to the complexified Kähler parameter and genus zero free energy determining the Gromov–Witten expansion (2.4). It turns out that the single logarithmic solution gives t, while the double logarithmic solution leads to the derivative of $F_0(t)$ (up to an overall factor). More precisely, we have

$$t = -\varpi_1(z), \qquad \partial_t F_0(t) = \frac{\varpi_2(z)}{6}. \tag{2.56}$$

Note that as $z \to 0$, we have $\mathrm{e}^{-t} \sim z \to 0$, and this is the large radius limit, as we mentioned before. From (2.56) we can compute the genus zero free energy as

$$F_0(t) = \frac{t^3}{18} + 3\mathrm{e}^{-t} - \frac{45}{8}\mathrm{e}^{-2t} + \frac{244}{9}\mathrm{e}^{-3t} - \frac{12333}{64}\mathrm{e}^{-4t} + \mathcal{O}(\mathrm{e}^{-5t}). \tag{2.57}$$

Exercise 2.5. Derive (2.57) from (2.53), (2.54) and (2.56). Verify the result with the topological vertex expansion in Exercise 2.2.

The choice of cycles above defines the large radius frame. Let us now consider the conifold frame. In order to construct it, we have to find an appropriate conifold coordinate. This must be a combination of periods vanishing at the conifold point and having good local properties there. In the case of local $\mathbb{P}^2$, this coordinate is given by

$$\lambda(z) = \frac{1}{4\pi}\left(\omega_c(z) - \pi^2\right), \tag{2.58}$$

where

$$\omega_c(z) = \log^2(-z) + 2\log(-z)\widetilde{\varpi}_1(z) + \widetilde{\varpi}_2(z). \tag{2.59}$$

The conifold frame is then defined by the relation

$$\begin{aligned} \frac{\partial F_0^c}{\partial \lambda} &= -\frac{2\pi}{3}t \pm \frac{2\pi^2 \mathrm{i}}{3}, \\ \lambda &= \frac{3}{2\pi}\partial_t F_0 \pm \frac{\mathrm{i}}{2\pi}t - \frac{\pi}{2}. \end{aligned} \tag{2.60}$$

The second relation is of course a consequence of (2.58). The choices of signs in (2.60) are correlated with the choice of branch cut of $\log(z)$ for $-1/27 < z < 0$, and they have been made in such a way that λ and $\partial_\lambda F_0^c$ are real in that interval. The genus zero free energy in the conifold frame can be computed explicitly and it has the form,

$$F_0^c(\lambda) = \frac{1}{2}\lambda^2\left(\log\left(\frac{\lambda}{3^{5/2}}\right) - \frac{3}{2}\right) - \frac{\lambda^3}{36\sqrt{3}} + \frac{\lambda^4}{7776} + \frac{7\lambda^5}{87480\sqrt{3}} - \frac{529\lambda^6}{62985600} + \mathcal{O}\left(\lambda^7\right). \tag{2.61}$$

This is precisely the universal behavior obtained in (2.25) for the genus zero free energy (we have chosen the normalization of λ so that it agrees with the canonical conifold coordinate, leading to the first equation in (2.25)). With some additional work, one can check that the higher genus free energies satisfy as well (2.25). As an example, the genus one and two free energies have the local expansion,

$$\begin{aligned} F_1^c(\lambda) &= -\frac{1}{12}\log(\lambda) + \frac{5\lambda}{72\sqrt{3}} - \frac{\lambda^2}{7776} - \frac{5\lambda^3}{17496\sqrt{3}} + \frac{283\lambda^4}{8398080} - \frac{43\lambda^5}{5668704\sqrt{3}} + \mathcal{O}\left(\lambda^6\right), \\ F_2^c(\lambda) &= -\frac{1}{240\lambda^2} + \frac{\lambda}{6480\sqrt{3}} - \frac{3187\lambda^2}{125971200} + \frac{239\lambda^3}{28343520\sqrt{3}} - \frac{19151\lambda^4}{28570268160} + \mathcal{O}\left(\lambda^5\right). \end{aligned} \tag{2.62}$$

As we will see later, we will be able to recover these series (and more) from a non-perturbative definition of the topological string on local $\mathbb{P}^2$.

3 Resurgence and topological strings

Since the sequence of topological string free energies $F_g(\mathbf{t})$ is factorially divergent, one can have a first handle on non-perturbative aspects by using the theory of resurgence (a short review of this theory can be found in the Appendix). This program was first proposed in [35, 41, 42], and it has experienced many interesting developments in recent years. The first thing we can ask is: what is the *resurgent structure* of the topological string? The resurgent structure, as we recall in the Appendix, is the collection of the Borel singularities of the Borel transform, together with the trans-series associated to them, and the Stokes constants. Modulo some assumptions on endless analytic continuation of the Borel transform, this is a well posed mathematical problem with a unique answer. The resurgent structure gives the collection of all non-perturbative sectors of the theory which can be obtained from the study of perturbation theory. From the point of view of physics, this gives candidate trans-series that complement the perturbative series and that can be used to obtain non-perturbative answers via Borel resummation. However, as we will see, the resurgent structure is interesting in itself, since the Stokes constants turn out to be, conjecturally, non-trivial invariants of the CY: the Donaldson–Thomas (DT) or BPS invariants.

3.1 Warm-up: the conifold and the resolved conifold

The problem of determining the resurgent structure of the topological string free energies is very difficult, due to the lack of explicit expressions for the genus g free energies. There are however two cases where one can determine this structure analytically: the conifold free energies (2.25), i.e. the leading singularities of the topological string free energy near a conifold singularity, and the free energies of the resolved conifold (2.23). These two examples were first worked out by S. Pasquetti and R. Schiappa in [43] and they turn out to be fundamental ingredients in the full theory.

Let us start with the conifold free energies. We consider the formal power series

$$\varphi(z) = \sum_{g\geq 2} a_{2g}\lambda^{2-2g}z^{2g-2}, \qquad a_{2g} = \frac{B_{2g}}{2g(2g-2)}. \tag{3.1}$$

It turns out to be more convenient to use the Borel transform (A.3). One finds,

$$\begin{aligned}\widetilde{\varphi}(\zeta) &= \sum_{g\geq 2}\frac{a_{2g}}{(2g-3)!}\zeta^{2g-3} = \sum_{g\geq 2}\frac{B_{2g}}{2g(2g-2)!}\lambda^{2-2g}\zeta^{2g-3}\\ &= \frac{1}{\zeta}\left\{-\frac{1}{12}+\frac{\lambda^2}{\zeta^2}-\frac{1}{4\sinh^2\left(\frac{\zeta}{2\lambda}\right)}\right\}.\end{aligned} \tag{3.2}$$

Note that $\widehat{\varphi}(\zeta)$ is a primitive of this function, and this is why it is difficult to obtain an explicit expression for it.

The singularities of the Borel transform occur at the imaginary axis, and they are located at

$$\zeta = 2\pi \mathrm{i}\ell\lambda, \qquad \ell\in\mathbb{Z}\backslash\{0\}. \tag{3.3}$$

They are double poles. Let us consider the Stokes ray going through the singularities with $\ell > 0$, at the angle $\theta = \pi/2$. The discontinuity of the lateral Borel resummations for that angle is simply computed by the sum of residues at those poles

$$
\begin{aligned}
s_{+}(\varphi)(z)-s_{-}(\varphi)(z) &= -2\pi\mathrm{i}\sum_{\ell=1}^{\infty}\mathrm{Res}_{\zeta=2\pi\mathrm{i}\ell\lambda}\left(\widetilde{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\right) \\
&= \frac{\mathrm{i}}{2\pi}\sum_{\ell\geq 1}\left\{\frac{1}{\ell}\left(\frac{\mathcal{A}}{z}\right)+\frac{1}{\ell^{2}}\right\}\mathrm{e}^{-\ell\mathcal{A}/z},
\end{aligned}
\tag{3.4}
$$

where

$$
\mathcal{A}=2\pi\mathrm{i}\lambda. \tag{3.5}
$$

If we consider the singularities in the negative imaginary axis, we find the same result, but with negative ℓ. By comparing the second line in (3.4) to (A.16), we can read the trans-series:

$$
\varphi_{\ell\mathcal{A}}(z)=\frac{1}{2\pi}\left\{\frac{1}{\ell}\left(\frac{\mathcal{A}}{z}\right)+\frac{1}{\ell^{2}}\right\}\mathrm{e}^{-\ell\mathcal{A}/z}, \qquad \ell\in\mathbb{Z}\backslash\{0\}. \tag{3.6}
$$

From now on we will focus on the trans-series for positive ℓ, although the results can be extended to the ones with negative ℓ. We will normalize these trans-series as in (3.6), therefore in (3.4) the corresponding Stokes constant is one. In more complicated CY manifolds there are amplitudes of the form (3.6) with non-trivial Stokes constants, as we will see in a moment.

From a physicist perspective, the second line of (3.4) looks like a sum over multi-instantons with action $\mathcal{A}$. We will refer to (3.6) as a *Pasquetti–Schiappa instanton amplitude*, with action $\mathcal{A}$. The expansion around each instanton is truncated at next-to-leading order in the coupling constant z (which should be identified with the string coupling constant g_s). The sum over $\ell > 0$ can be performed in closed form and one finds,

$$
\left(\mathfrak{S}_{\pi/2}-1\right)(\varphi)=\frac{\mathrm{i}}{2\pi}\left\{\mathrm{Li}_{2}\left(\mathrm{e}^{-\mathcal{A}/z}\right)-\frac{\mathcal{A}}{z}\log\left(1-\mathrm{e}^{-\mathcal{A}/z}\right)\right\}=\log\Phi_{1}\left(-\frac{\mathcal{A}}{2\pi z}\right). \tag{3.7}
$$

In the last step we have used (B.12) to identify this function as Faddeev's non-compact quantum dilogarithm $\Phi_{\mathsf{b}}(x)$, evaluated at $\mathsf{b}=1$. Faddeev's quantum dilogarithm is a remarkable special function introduced in [44], which appears in many contexts in modern mathematical physics. In the Appendix B we list some of its properties, which will be also useful in section 4. Since $\mathfrak{S}_{\pi/2}$ is an automorphism its action on $Z_{\mathrm{con}}=\mathrm{e}^{\varphi}$ is given by the multiplicative action

$$
\mathfrak{S}_{\pi/2}\left(Z_{\mathrm{con}}\right)=\Phi_{1}\left(-\frac{\mathcal{A}}{2\pi z}\right)Z_{\mathrm{con}}. \tag{3.8}
$$

Remark 3.1. Writing the Stokes automorphism in terms of Faddeev's quantum dilogarithm allows for a compact notation, but there is a deeper reason for it. In the local case, the topological string admits a deformation or refinement by using the so-called Omega background [30, 45]. This deformation can be parametrized by a complex number b, and the undeformed or unrefined case corresponds to the value $\mathsf{b}=1$. It turns out that the formula (3.7) admits a generalization to the refined a case, in which the r.h.s. involves Faddeev's dilogarithm for arbitrary b, see [46] for the details.

As we mentioned above, the results for the conifold are very useful, and make it possible to obtain the trans-series for the resolved conifold immediately. The reason is that, thanks to (2.24), we can write the resolved conifold free energies as an infinite sum of conifold free energies:

$$
F_{g}(t)=\sum_{m\in\mathbb{Z}}\frac{B_{2g}}{2g(2g-2)}\left(\mathrm{i}t+2\pi m\right)^{2-2g}, \qquad g\geq 2. \tag{3.9}
$$

The trans-series are again of the Pasquetti–Schiappa form, but now the action is labelled by an additional integer number m, and is given by

$$\mathcal{A}_m = 2\pi t + 4\pi^2 \mathrm{i} m, \qquad m \in \mathbb{Z}. \tag{3.10}$$

There is a simple extension of this result which is also useful. Let us consider the expression (2.15) for $g \geq 2$, which is valid near the large radius point $\mathrm{Re}(t_i) \gg 1$. The first term in the parentheses is identical to the free energy for the resolved conifold, and it is easy to see that it is the only term growing factorially. Therefore, we expect that, close enough to the large radius point, we will have a sequence of Borel singularities at

$$\mathcal{A}_{\mathbf{d},m} = 2\pi \mathbf{d} \cdot \mathbf{t} + 4\pi^2 \mathrm{i} m, \qquad m \in \mathbb{Z}, \tag{3.11}$$

where $\mathbf{d}$ are the values of the degrees which lead to a non-zero Gopakumar–Vafa invariant $n_0^{\mathbf{d}}$. The trans-series associated to these singularities are of the form

$$n_0^{\mathbf{d}}\, \varphi_{\ell \mathcal{A}_{\mathbf{d},m}}(g_s). \tag{3.12}$$

Therefore, $n_0^{\mathbf{d}}$ (which is an integer) has to be interpreted as the Stokes constant associated to the sequence of singularities (3.11), and the corresponding Stokes automorphism can be written as

$$\mathfrak{S}\left(Z_{\mathrm{LR}}\right) = \left[\Phi_1\left(-\frac{\mathcal{A}}{2\pi z}\right)\right]^{n_0^{\mathbf{d}}} Z_{\mathrm{LR}}, \tag{3.13}$$

where $\mathcal{A}$ is given by (3.11), and we have denoted the partition function in the large radius frame by Z_{LR}.

Exercise 3.2. Let us consider the constant map contribution (2.7). Show that it can be written as

$$c_g = -\frac{B_{2g}}{2g(2g-2)} \sum_{m=1}^{\infty} (2\pi m)^{2-2g} \tag{3.14}$$

Deduce that, if

$$\varphi(z) = \sum_{g\geq 2} c_g z^{2g-2}, \tag{3.15}$$

one has the discontinuity formula

$$s_+(\varphi)(z) - s_-(\varphi)(z) = -\frac{\mathrm{i}}{2\pi} \sum_{\ell\geq 1} \sigma(\ell) \left\{ \frac{1}{\ell}\left(\frac{\mathcal{A}}{z}\right) + \frac{1}{\ell^2} \right\} \mathrm{e}^{-\ell \mathcal{A}/z}, \tag{3.16}$$

where

$$\mathcal{A} = 4\pi^2 \mathrm{i}, \qquad \sigma(\ell) = \sum_{m|\ell} \left(\frac{\ell}{m}\right)^2. \tag{3.17}$$

This result was derived in [47] with a different technique. □

3.2 The general multi-instanton trans-series

In the examples of Stokes automorphisms considered so far, the location of the Borel singularities for the free energies or partition function had the following property: $\mathcal{A}$ was given by the flat coordinate of the frame in which we were computing the partition function (up to a shift by a constant). However, we expect $\mathcal{A}$ to be given by a linear combination of periods of the CY. In other words, and restricting ourselves to the local case with $g_\Sigma = 1$ for simplicity, we expect to have

$$\mathcal{A} = c\frac{\partial F_0}{\partial t} + dt + 4\pi^2 \mathrm{i} m, \qquad m \in \mathbb{Z}. \tag{3.18}$$

This principle was first stated in [48], based on previous insights on instantons in matrix models [49]. Additional arguments and evidence for this principle were given in [39, 50]. In addition it was emphasized in [47] that, with appropriate normalizations, Borel singularities are *integer* linear combinations of periods. This means that c and d in (3.18) are universal constants, times integers.

Let us now give a simple argument for obtaining the trans-series associated to a more general Borel singularity, of the form (3.18) with $c \neq 0$. Let us suppose that there is a frame where $\mathcal{A}$ is the flat coordinate $\tilde{t}$. This frame is called in [46, 47] an $\mathcal{A}$-frame. We will also assume that in this frame the Stokes automorphism is purely multiplicative,

$$\mathfrak{S}(Z(\tilde{t})) = \exp\left[\frac{\mathrm{i}\Omega}{2\pi}\left(\mathrm{Li}_2\left(\mathrm{e}^{-\mathcal{A}/g_s}\right) - \frac{\mathcal{A}}{g_s}\log\left(1-\mathrm{e}^{-\mathcal{A}/g_s}\right)\right)\right] Z(\tilde{t}). \tag{3.19}$$

Here, Ω is an Stokes constant. We know from (3.7) that this formula is true for the Borel singularity at the conifold frame, with $\Omega = 1$, and for the Borel singularities (3.11) at large radius, with $\Omega = n_0^{\mathbf{d}}$ (additional cases where this happens are discussed in [47].) The general $\mathcal{A}$-frame is defined by the transformation

$$\begin{pmatrix} \partial_{\tilde{t}}\widetilde{F}_0 \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} A & B \\ c & d \end{pmatrix}\begin{pmatrix} \partial_t F_0 \\ t \end{pmatrix} + \begin{pmatrix} \sigma \\ 4\pi^2 \mathrm{i} m \end{pmatrix}, \tag{3.20}$$

where A, B are such that $Ad - Bc = 1$, and σ is a shift. We can now invert this transformation and use the general formula (2.42), to find

$$S(\tilde{t}, t) = -\frac{t\tilde{t}}{c} + \frac{d}{2c}t^2 + \frac{4\pi^2 \mathrm{i} m}{c}t + s(\tilde{t}), \tag{3.21}$$

where

$$s(\tilde{t}) = \frac{A}{2c}\tilde{t}^2 + \left(\sigma - 4\pi^2\mathrm{i} m\frac{A}{c}\right)\tilde{t}. \tag{3.22}$$

The partition functions are then related by (2.43). We note that we can write

$$Z(t) = \exp\left(-\frac{d}{2cg_s^2}t^2 - \frac{4\pi^2\mathrm{i}m}{cg_s^2}t\right)\widehat{Z}(t), \tag{3.23}$$

where

$$\widehat{Z}(t) = \int Z(\tilde{t})\exp\left(\frac{t\tilde{t}}{cg_s^2} + \frac{s(\tilde{t})}{g_s^2}\right)\mathrm{d}\tilde{t}. \tag{3.24}$$

We will now assume that the Stokes automorphism acting on $Z(t)$ is obtained as a Fourier transform of the Stokes automorphism acting on $Z(\tilde{t})$. This is very natural, since the action on the Stokes automorphism can be regarded as a trans-series generalization of the perturbative partition function, and the Fourier transform acting on the perturbative sector extends naturally to the full trans-series. We then have

$$\mathfrak{S}(Z(t)) = \int \mathfrak{S}(Z(\tilde{t}))\mathrm{e}^{-S(t,\tilde{t})/g_s^2}\mathrm{d}\tilde{t} = \exp\left(-\frac{d}{2cg_s^2}t^2 - \frac{4\pi^2\mathrm{i}m}{cg_s^2}t\right)\left[\Phi_1\left(-\frac{g_s c}{2\pi}\partial_t\right)\right]^{\Omega}\widehat{Z}(t). \tag{3.25}$$

We have used here the standard property that insertions of $\tilde{t}$ inside the Fourier transform can be traded by derivatives. We can also write this as

$$\mathfrak{S}(\widehat{Z}(t)) = \left[\Phi_1\left(-\frac{g_s c}{2\pi}\partial_t\right)\right]^{\Omega}\widehat{Z}(t). \tag{3.26}$$

This formula was derived in [47, 51, 52] by using a more complicated method based on the holomorphic anomaly equations of [20], which has the advantage that it applies to compact CY manifolds as well. The derivation presented here, in a slightly less general form, can be found in [53].

We can write (3.26) more explicitly as

$$\mathfrak{S}_{\mathcal{C}}(\widehat{Z}(t)) = \exp\left[\frac{\mathrm{i}\Omega}{2\pi}\left(\mathrm{Li}_2\left(\mathcal{C}\mathrm{e}^{-cg_s\partial_t}\right) - cg_s\partial_t\log\left(1-\mathcal{C}\mathrm{e}^{-cg_s\partial_t}\right)\right)\right]\widehat{Z}(t), \tag{3.27}$$

where we have introduced a parameter $\mathcal{C}$ to keep track of the order of the exponential corrections. By expanding the r.h.s. of this equation in powers of $\mathcal{C}$ we find

$$\widehat{Z}(t) + \frac{\mathrm{i}\Omega}{2\pi}\mathcal{C}\left(1 + cg_s\partial_t\widehat{F}(t-cg_s;g_s)\right)\mathrm{e}^{\widehat{F}(t-cg_s;g_s)} + \cdots \tag{3.28}$$

We can introduce the multi-instanton sectors of the free energy as

$$\sum_{\ell\geq 0}\mathcal{C}^{\ell}F^{(\ell)}(t;g_s) = -\mathrm{i}\log\left(\mathfrak{S}_{\mathcal{C}}(Z(t))\right), \tag{3.29}$$

and we find that the first instanton sector is

$$F^{(1)}(t;g_s) = \frac{\Omega}{2\pi}\left(1 + cg_s\partial_t\widehat{F}(t-cg_s;g_s)\right)\mathrm{e}^{\widehat{F}(t-cg_s;g_s)-\widehat{F}(t;g_s)}. \tag{3.30}$$

Higher instanton sectors can be obtained in a straightforward way. Let us make some comments on the structure of this formula. First of all, the total free energy $\widehat{F}(t;g_s)$ differs from $F(t;g_s)$ only in its genus zero piece, i.e. we have

$$\widehat{F}_0(t) = F_0(t) + \frac{d}{2c}t^2 + \frac{4\pi^2\mathrm{i}m}{c}t, \tag{3.31}$$

and is such that

$$\mathcal{A} = c\frac{\partial\widehat{F}_0(t)}{\partial t}. \tag{3.32}$$

In particular, the exponential factor in (3.30) has the g_s expansion

$$\exp\left(\widehat{F}(t-cg_s;g_s) - \widehat{F}(t;g_s)\right) = \mathrm{e}^{-\mathcal{A}/g_s}\left(1+\mathcal{O}(g_s)\right), \tag{3.33}$$

which indicates the non-perturbative character of (3.30). The exponent in (3.33) can be interpreted as the difference between the free energies of two different backgrounds, represented by two different moduli of the CY: the background t of the perturbative sector, and the background $t - cg_s$, in which we shift t. It is easy to see that in the ℓ-th instanton sector the shift is given by $t - \ell c g_s$. Since, with the appropriate normalizations, c is an integer, this suggests that t is "quantized" in units of g_s. Such a quantization is typical of topological string theories described by matrix models or field theories, in which the CY modulus is interpreted as a 't Hooft parameter and has the form $N g_s$, where N is the rank of the matrix model [8, 10, 12]. We will elaborate on this in section 4.

The result above for the Stokes automorphism is valid for local CY manifolds with a single modulus, but it has an obvious generalization to general CYs [52]. In the general case, the analogue of the expression (3.18) for the location of a Borel singularity is simply

$$\mathcal{A} = c^I \mathcal{F}_I + d_I X^I, \tag{3.34}$$

where summation over the repeated indices is understood. If all $c^I = 0$, the Stokes automorphism is given by the formula (3.19). If not all c^I vanish, one first defines a new genus zero free energy by

$$\mathcal{A} = c^I \frac{\partial \widehat{F}_0}{\partial X^I}, \tag{3.35}$$

as in the local case. It can be written as

$$\widehat{F}_0(X^I) = F_0(X^I) + \frac{1}{2} a_{IJ} X^I X^J, \qquad a_{IJ} c^I = d_J, \tag{3.36}$$

which is the counterpart of (3.31) (the final formulae will not depend on the choice of a_{IJ}, but only on c^I, d_J.) One also has to define a new genus one free energy

$$\widehat{F}_1 = F_1 - \left(\frac{\chi}{24} - 1 \right) \log X^0, \tag{3.37}$$

where we recall that χ is the Euler characteristic of the CY M. This is a new ingredient in the compact case which was found in [47]. The redefinitions of the genus zero and one free energies lead to a new total free energy which will be denoted by $\widehat{F}(X^I; g_s)$. It is given by

$$\widehat{F}(X^I; g_s) = g_s^{-2} \widehat{F}_0(X^I) + \widehat{F}_1(X^I) + \sum_{g \geq 2} g_s^{2g-2} F_g(X^I). \tag{3.38}$$

Then, one has the following generalization of (3.27),

$$\mathfrak{S}_{\mathcal{C}}(\widehat{Z}) = \exp \left[\frac{\mathrm{i}\Omega}{2\pi} \left(\mathrm{Li}_2 \left(\mathcal{C} \mathrm{e}^{-g_s c^I \partial_I} \right) - g_s \log \left(1 - \mathcal{C} \mathrm{e}^{-g_s c^I \partial_I} \right) c^I \partial_I \right) \right] \widehat{Z}, \tag{3.39}$$

where we have denoted $\widehat{Z} = \mathrm{e}^{\widehat{F}}$, and

$$\partial_I = \frac{\partial}{\partial X^I}. \tag{3.40}$$

3.3 BPS states and the resurgent structure of topological strings

We have now general results for the trans-series associated to the different Borel singularities, but we still need to know the precise location of the singularities and the corresponding Stokes constants.

The positions of the Borel singularities for the topological string free energies depend on the value of the moduli of the CY. As we move in moduli space, the singularities change their position and sometimes change *discontinuously*. This phenomenon was first observed in [54], in the resurgent structure of quantum of WKB periods associated to the Schrödinger equation. This discontinuous change will be referred to as *wall-crossing*, since it is indeed related to wall-crossing phenomena for BPS states in supersymmetric gauge theory [55–57], as reviewed in A. Neitzke lectures, and in the theory of Donaldson–Thomas invariants [58].

In the theory of BPS states or Donaldson–Thomas invariants on a CY threefold, the BPS states are characterized by a charge $\boldsymbol{\gamma} \in \Gamma$, where Γ is an appropriate lattice. For example, for a compact CY threefold M in the A-model one has

$$\Gamma = H^{\mathrm{ev}}(M, \mathbb{Z}), \tag{3.41}$$

and its rank is $2(s+1)$, where we recall that $s = h^{1,2}(M)$. If we choose a basis for this lattice we can write $\boldsymbol{\gamma}$ in terms of two pairs of vectors of rank $s+1$ with integer entries, $\boldsymbol{\gamma} = (c^I, d_I)$, where $I = 0, 1, \cdots, s$. We can think of c^0, d_0 as D6 and D0 brane charges, respectively, and of c^a, d_a, $a = 1, \cdots, s$, as D4 and D2 brane charges. The central charge corresponding to such an element of Γ is given by

$$Z_{\boldsymbol{\gamma}} = c^I \mathcal{F}_I + d_I X^I, \tag{3.42}$$

where summation over the repeated indices is understood. Let us note that, in the case of toric CY manifolds, D6 branes decouple, and the charge $\boldsymbol{\gamma}$ is specified by $2s+1$ integers which we will denote by c^a, d_a and m, with $a = 1, \cdots, s$. The central charge reads then

$$Z_{\boldsymbol{\gamma}} = c^a \frac{\partial F_0}{\partial t^a} + d_a t^a + 4\pi^2 \mathrm{i} m. \tag{3.43}$$

Given a point in moduli space, one can define BPS or DT invariants associated to a charge $\boldsymbol{\gamma}$, which we will denote by $\Omega_{\boldsymbol{\gamma}}$. The spectrum of BPS states is the set of charges $\boldsymbol{\gamma}$ for which $\Omega_{\boldsymbol{\gamma}} \neq 0$.

A very simple example of BPS spectrum and invariants occurs in Seiberg–Witten (SW) theory [55]. In this framework, the theory is associated to a local CY manifold described by a curve

$$y^2 + 2\cosh(x) = 2u. \tag{3.44}$$

We note that the variable x appears in exponentiated form, but not the variable y. The moduli space is parametrized by the complex number u (this is the famous u-plane of SW theory), and there are two independent periods which can be chosen to be

$$\begin{aligned} a(u) &= \frac{2\sqrt{2}}{\pi}\sqrt{u+1}E\left(\frac{2}{u+1}\right), \\ a_D(u) &= \frac{\mathrm{i}}{2}(u-1)_2F_1\left(\frac{1}{1}, \frac{1}{2}, 2; \frac{1-u}{2}\right). \end{aligned} \tag{3.45}$$

The lattice of charges here has rank two, and we will denote

$$\boldsymbol{\gamma} = (\gamma_e, \gamma_m), \tag{3.46}$$

where $\gamma_{e,m}$ refers to the electric (respectively, magenetic) charge. Then, the central charge is given by

$$Z_{\boldsymbol{\gamma}}(u) = 2\pi\left(\gamma_e a(u) + \gamma_m a_D(u)\right), \tag{3.47}$$

where we have introduced an appropriate normalization factor 2π for the periods. The spectrum of BPS states in this theory has been investigated intensively, see e.g. [55–57, 59], and has been

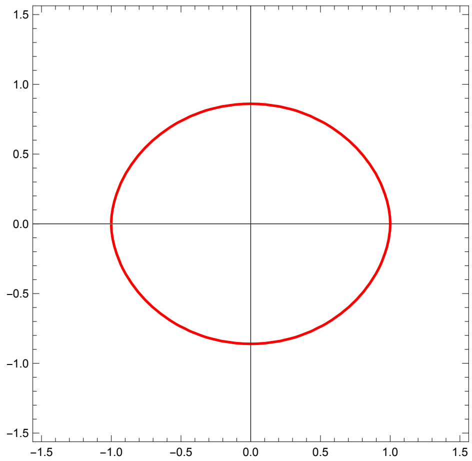

Figure 3. The curve of marginal stability in the u-plane, defined by the equation (3.48).

described in detail in the lectures by A. Neitzke in this school. First of all, the spectrum depends on the value of the modulus u. Inside the so-called *curve of marginal stability*, defined by

$$\mathrm{Im}\left(\frac{a_D}{a}\right) = 0, \tag{3.48}$$

we have the so-called strong coupling spectrum: the only stable states have charges $\boldsymbol{\gamma}_M = (0,1)$ (a magnetic monopole) and $\boldsymbol{\gamma}_D = (1,1)$ (a dyon). Outside this curve, we have the so-called weak coupling spectrum, consisting of a W-boson and a tower of dyons, with charges, respectively,

$$\boldsymbol{\gamma}_W = (1,0), \qquad \boldsymbol{\gamma}_n = (n,1), \quad n \in \mathbb{Z}. \tag{3.49}$$

See Fig. 3 for a plot of the curve of marginal stability in the u-plane. States with charges $-\boldsymbol{\gamma}$ also belong to the spectrum. The corresponding DT invariants have the values

$$\Omega_{(0,1)} = \Omega_{(1,1)} = 1, \tag{3.50}$$

in the strong coupling region, while in the weak coupling region we have

$$\Omega_{(n,1)} = 1, \qquad \Omega_{(1,0)} = -2. \tag{3.51}$$

We can now propose the following description of the resurgent structure of the topological string free energies.

1. The total topological string free energy in a given frame is a resurgent function. Its Borel singularities are integer linear combinations of the CY periods, as in (3.34). We note that these singularities are determined by a charge vector $\boldsymbol{\gamma}$, and their location is given by the central charge (3.42) of a BPS state with the same charge vector (up to a normalization).

2. The singularities display a multi-covering structure: given a singularity $\mathcal{A}$, all its integer multiples $\ell\mathcal{A}$, $\ell \in \mathbb{Z}\backslash\{0\}$, appear as singularities as well. The Stokes automorphism for the singularities occurring along a half-ray $\ell\mathcal{A}$, $\ell \in \mathbb{Z}_{>0}$ is given by (3.19) (for the case in which all $c^I = 0$) or (3.39) (for the case in which not all c^I vanish).

3. The Stokes constant Ω appearing in these Stokes automorphisms is the BPS or DT invariant Ω_{γ} associated to the BPS state with central charge $\mathcal{A}$.

This conjecture concerns the resurgent structure of the topological string free energies in a fixed frame. However, it follows from the description that the Borel singularities and Stokes constant do *not* depend on the frame. Let us clarify this point. The free energies in a given frame F_g are not globally defined on the moduli space, and they are analytic only on a region, typically centered around a special point, like the large radius point or the conifold point. At a point in the moduli space where the free energies in two different frames are well-defined, they have conjecturally the *same* resurgent structure, determined by the BPS structure at that point. However, the form of the Stokes automorphism might be different, depending on whether the frame is an $\mathcal{A}$-frame or not.

Note that this picture is natural from the point of view of the generalized Fourier transform: we can gather the information on the resurgent structure in a trans-series, i.e. in a collection of non-perturbative corrections to the perturbative partition function. The Stokes constants are coefficients in this trans-series, and the location of the Borel singularities can be read from the exponentially small terms in g_s. Under a change of frame, the full trans-series transforms under a generalized Fourier transform. This does not change the coefficients of the trans-series, nor the exponentially small terms in g_s, as we saw in e.g. (3.33).

This conjecture above was built up in various works. The construction of explicit trans-series was started in [43] for the conifold and the resolved conifold. It was extended to the local case in [39, 50, 51], and to the general case in [47]. A compact formula for the Stokes automorphism based on these developments was worked out in [52]. The connection between Stokes constants and BPS invariants was anticipated in [60], and stimulated by a similar connection discovered in complex Chern–Simons theory in [61–63]. A first formulation of this conjecture was presented in [64], which was later clarified in [53] in the light of the conjecture above. The conjecture was noted to hold for the resolved conifold in [65–67], and extended to more general cases in [47, 51]. The formulation above can be found in [46, 52, 53] (in addition, [46] the conjecture is generalized to the refined topological string).

There is both direct and indirect evidence for this conjecture.

1. One first piece of direct evidence is the following. As shown in (3.11), (3.13), Borel singularities with charges $\boldsymbol{\gamma} = (0, d_I)$, therefore corresponding to D2-D0 bound states, appear at large radius. Their Stokes constants are given by $n_0^{\mathbf{d}}$, where $\mathbf{d} = (d_1, \cdots, d_s)$ are the D2 charges. This is precisely the DT invariant for a D2-D0 BPS state (see e.g. [68]). In addition, the behavior of the free energies in the conifold frame near the conifold locus indicates that in this region there is a D4-D0 bound state with DT invariant equal to 1. This is also expected, since the conifold behavior of the free energies that we have considered is due to a single hypermultiplet becoming massless at the locus [69, 70].

2. Indirect evidence for the conjecture comes from comparison with a different line of work, studying the geometry of the hypermultiplet moduli space in CY compactifications (see [68] for a review). One of the things which is found in this study is that there is a natural action of the so-called Kontsevich–Soibelman automorphisms on the topological string partition function [71, 72]. This action turns out to be identical to the Stokes automorphism that we have just described, *provided* the Stokes constants are identified with DT invariants.

3. There is additional indirect evidence for the above conjecture for local CY manifolds. In the local case one can consider WKB, or quantum periods associated to the quantum

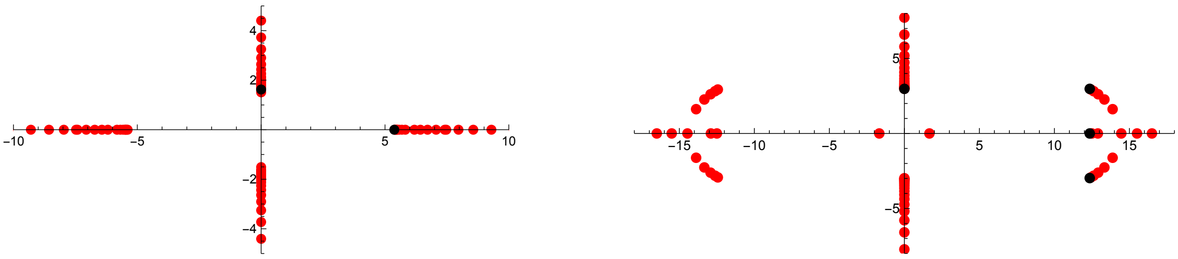

Figure 4. The Borel singularities of $\mathcal{F}(t)$ in Seiber–Witten theory, inside the curve of marginal stability with $u = 1/2$ (left), and outside the curve with $u = 5/2$ (right). The black dots in the figure in the left occur at the values $2\pi a_D$, $2\pi(a + a_D)$, and corresponds to the monopole and dyon state, respectively. The black dots in the figure in the right occur at the value $2\pi a$ (on the positive real axis), corresponding to the W-boson, at the value $2\pi a_D$ (on the imaginary axis), corresponding to the manopole, and at the values $2\pi(a \pm a_D)$ (above and below the positive real axis, respectively). The latter correspond to the dyons $(1, \pm 1)$, which are the first states in the tower.

version of the curve (2.33), in which x, y are promoted to canonically commuting Heisenberg operators (we will come back to this subject in section 4). These periods define a different topological string theory, usually called the *Nekrasov–Shatashvili (NS) topological string* [6]. The quantum periods associated to the quantum mirror curve are also factorially divergent power series, and one can study their resurgent structure (see e.g. [73] for references to the extensive literature on the subject). In [74] it was argued that, in the local case, the Stokes constants appearing in the resurgent structure of the standard topological string are the same ones appearing in the resurgent structure of the quantum periods. The latter should be directly related to DT invariants, as expected from the 4d results of [57, 75, 76].

Additional evidence for the conjecture comes from direct comparisons between calculations of Stokes constants in the resurgent structure of the topological string, and calculations of BPS invariants. As we saw above, one of the simplest examples of a BPS spectrum and invariants is the one appearing in SW theory, which displays already a non trivial wall-crossing structure. One can associate to this theory a sequence of topological string free energies $\mathcal{F}_g$, $g \geq 0$, in many different ways, e.g. by considering topological recursion as applied to the SW curve (3.44). Can we match the BPS structure to the resurgent structure of these topological string free energies? This was answered in the affirmative in [53], by a numerical study of the sequence of free energies $\mathcal{F}_g$. It is convenient to do this in the so-called magnetic frame, in which the flat coordinate is chosen to be $t = -\mathrm{i} a_D(u)$ (we recall that $a_D(u)$ is defined in (3.45)). If the conjecture above is correct, we should find a very different structure of singularities in the Borel plane depending on whether u is inside or outside the curve of marginal stability. Inside the curve, we should find two Borel singularities (together with their reflections), corresponding to the monopole and dyon states. Outside the curve, we should find the monopole, the W particle, and a tower of dyons. This is precisely what is obtained in a numerical analysis, as shown in Fig. 4. Since this is a numerical approximation, we only see the dyons in the tower whose central charges are closer to the origin. In addition, a detailed numerical calculation confirms the values of the DT invariants (3.50), (3.51).

4 Topological strings from quantum mechanics

In this final section we will explore the question of finding a non-perturbative completion of the topological string, i.e. of finding a well-defined function whose asymptotics reproduces the perturbative free energy as its asymptotic series. Let us note that, under some mild assumptions, one can obtain non-perturbative completions by just considering (lateral) Borel resummations of the asymptotic series. One can enrich this simple completion by adding trans-series. Since general trans-series have arbitrary coefficients, the resurgent analysis of the previous section gives an infinite family of completions. Reality constraints can restrict the values of these coefficients, but it is clear that we need some additional physical input in order to make progress and select a specific non-perturbative completion.

In physical theories, the ultimate arbiter on the correct non-perturbative completion should be comparison to experiment. In topological string theory we don't have such an arbiter, so we can indulge in being guided by beauty, without incurring the risk of being harangued by Sabine Hossenfelder on how to do good science [77]. In this section we will consider a non-perturbative completion which is mathematically non-trivial and beautiful. It is motivated by deep physical insights, related to large N dualities and to the quantization of geometry.

4.1 Warm-up: the Gopakumar–Vafa duality

Perhaps the simplest non-perturbative completion of a topological string free energy is the GV duality between the resolved conifold and Chern–Simons theory on the three-sphere [10]. It displays some of the properties of the more general non-perturbative completion that we will introduce in this section, so we will present a brief summary. A more complete treatment can be found in [15, 78].

The inspiration for [10] came from large N dualities between gauge theories and string theories. This is an old idea that goes back to the work of 't Hooft on the $1/N$ expansion and was later implemented in the AdS/CFT correspondence. According to these dualities, a gauge theory with gauge group $U(N)$ and gauge coupling constant g_s is equivalent to a string theory with the same coupling constant and a geometric modulus proportional to the so-called *'t Hooft coupling* of the gauge theory, $g_s N$. Since the modulus is given by a positive integer, times the coupling constant, it is in a sense "quantized" (a similar phenomenon was already noted in the formula (3.30)). We can consider the so-called *'t Hooft limit*

$$N \to \infty, \qquad g_s \to 0, \qquad N g_s \text{ fixed.} \tag{4.1}$$

As N becomes large, the discreteness of the 't Hooft parameter should become inessential. More precisely, we expect that in the 't Hooft limit a geometric, continuous description of this parameter will emerge, so that we can identify it with a modulus. We also expect that the observables of the theory will have an asymptotic expansion in inverse powers of $1/N$, or $1/N$ expansion. For example, in the case of the the vacuum free energy of the gauge theory, we have the $1/N$ expansion

$$F(g_s, N) \sim \sum_{g \geq 0} F_g(t) g_s^{2g-2}. \tag{4.2}$$

The quantities $F_g(t)$ are interpreted as genus g free energies in a string theory.

In [10], Gopakumar and Vafa considered $U(N)$ Chern–Simons theory, a topological field theory in three dimensions studied and essentially solved by Witten in [79]. Witten found in particular a closed formula for the free energy of this theory on the three-sphere $\mathbb{S}^3$, which reads,

$$Z^{\rm CS}(g_s,N) = \left(\frac{g_s}{2\pi}\right)^{N/2} \prod_{j=1}^{N-1} \left(2\sin\frac{g_s j}{2}\right)^{N-j}. \tag{4.3}$$

A relatively simple computation done in [10] and reviewed in [15] shows that this free energy has an asymptotic expansion of the form (4.2), with

$$\begin{aligned} F_0^{\rm CS}(t) &= -\frac{t^3}{12} + \frac{\pi {\rm i}}{4} t^2 + \frac{\pi^2 t}{6} - \zeta(3) + {\rm Li}_3({\rm e}^{-t}), \\ F_1^{\rm CS}(t) &= -\frac{t-\pi {\rm i}}{24} + \zeta'(-1) + \frac{1}{12}\log g_s + \frac{1}{12}{\rm Li}_1({\rm e}^{-t}), \\ F_g^{\rm CS}(t) &= 2c_g + \frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}{\rm Li}_{3-2g}({\rm e}^{-t}), \qquad g \geq 2. \end{aligned} \tag{4.4}$$

In these equations,

$$t = {\rm i} g_s N, \tag{4.5}$$

and c_g is given in (2.7). These are the free energies of the resolved conifold (2.23), up to a polynomial piece in t which can be regarded as the perturbative part of the free energy. Therefore, at least at the level of free energies, there is a large N duality between topological strings on the resolved conifold, and $U(N)$ Chern–Simons gauge theory on the three-sphere.

Let us note that the exact CS free energy is a function of a positive integer N and of a parameter g_s. In the original CS theory, g_s is real and is related to the so-called CS level $k \in \mathbb{Z}$ by the equation

$$g_s = \frac{2\pi}{k+N}. \tag{4.6}$$

The expression (4.3) makes sense in principle for any complex value of g_s, although the free energy is singular for some special values of g_s. On the other hand, in the topological string side, the 't Hooft parameter is identified with a complexified Kähler parameter and it can be any complex number, as long as ${\rm Re}(t) > 0$ (in fact, one can consider more general values of t by a so-called flop transition to a different CY phase). The non-perturbative completion provided by the gauge theory is in principle restricted to values of g_s of the form (4.6), and values of t for which $t/g_s = {\rm i}N$. This is sometimes interpreted as saying that the continuous or geometric description "emerges" in the large N limit, out of a microscopic description which is not geometric -somewhat similarly to the continuous or fluid description of a many-particle system in the thermodynamic limit.

Another interesting observation is that the partition function (4.3) can be regarded as a matrix integral

$$Z^{\rm CS}(g_s,N) = \frac{{\rm e}^{-\frac{\hbar}{12}N(N^2-1)}}{N!} \int \prod_{i=1}^N \frac{{\rm d}u_i}{2\pi}\, {\rm e}^{-\frac{1}{2\hbar}\sum_{i=1}^N u_i^2} \prod_{i<j} \left(2\sinh\frac{u_i-u_j}{2}\right)^2, \tag{4.7}$$

where $\hbar = {\rm i} g_s$. This can be regarded as a deformation of the Gaussian matrix model, in which the standard Vandermonde interaction between eigenvalues gets deformed to a sinh interaction. This type of matrix models appears naturally in the context of localization of three-dimensional supersymmetric field theories [80] (see [81] for a review).

The Gopakumar–Vafa duality has a natural generalization by performing a quotient of both sides by an ADE discrete group Γ. In the case of $\Gamma = \mathbb{Z}_p$, one finds a duality between CS theory on the lens space $L(p, 1)$, and topological string theory on certain CY geometries which engineer $SU(p)$ gauge theories. This duality has been verified in detail when $p = 2$ [82], see [83] for additional verifications and generalizations.

One important aspect of these dualities is that they provide a reconstruction of perturbative topological string theory as the $1/N$ expansion of a gauge/matrix model system, but *without taking a double-scaling limit*, as it happens in the matrix models of non-critical strings. Double-scaled Hermitian matrix models are often unstable, since the scaling point corresponds to a point where the matrix model is not well-defined non-perturbatively in a straightforward way (for example, for pure 2d gravity, the double-scaling limit in terms of a quartic matrix model requires that the coupling of the quartic term is negative). The matrix model (4.7) is in contrast perfectly well-defined.

However, in its current form, these Gopakumar–Vafa-like dualities apply only to special non-compact CY geometries. We will now consider a different duality that holds conjecturally for *any* toric CY: the *topological string/spectral theory (TS/ST) correspondence*. In this correspondence, the dual of the string theory is not a field theory, but a one-dimensional quantum-mechanical model. We will end up however with matrix model representations for the topological string partition function, similar to (4.7). These matrix models will involve the usual 't Hooft limit, without double-scaling, and will be stable due to a remarkable property of the underlying quantum mechanics.

4.2 Quantum mirror curves and non-perturbative partition functions

The TS/ST correspondence was originally triggered by the observation that the genus zero free energy of a local CY manifold looks like a leading order WKB computation for a quantum system whose classical phase space is defined by the mirror curve (2.33). This suggests that the higher genus corrections might be understood as the result of an appropriate quantization of this curve [5].

What does "quantization of the curve" means? Since the natural Liouville form on the phase space (2.33) is (2.34), the natural way to to proceed is to promote x, y to canonically conjugate Heisenberg operators x, y, satisfying the commutation relation

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar. \tag{4.8}$$

Since mirror curves involve exponentiated variables, one has to consider the so-called Weyl operators $\mathrm{e}^{a\mathsf{x}}$, $\mathrm{e}^{b\mathsf{y}}$ obtained by exponentiation of Heisenberg operators. The equation for a mirror curve is a linear combination of terms of the form $\exp(ax + by)$, and as it is well-known the quantization of such a term suffers from ordering ambiguities. These can however be fixed fixed by using Weyl's prescription,

$$\mathrm{e}^{ax+by} \to \mathrm{e}^{a\mathsf{x}+b\mathsf{y}}. \tag{4.9}$$

We can therefore try to promote the equation of the curve to an operator, but this is in principle ill-defined since the equation is invariant under multiplication by $\exp(\lambda x+\mu y)$, with λ, μ arbitrary. To fix this new ambiguity, we have to write the curve in a canonical form, which in the case of mirror curves of genus one is given by

$$P_X(\mathrm{e}^x, \mathrm{e}^y) = \sum_j c_j \mathrm{e}^{a_j x + b_j y} + \kappa = 0, \tag{4.10}$$

where κ is the modulus. This is the form that we have used in e.g. (2.45). We now define the operator associated to the toric CY by

$$\mathsf{O}_X = \sum_j c_j \mathrm{e}^{a_j \mathsf{x} + b_j \mathsf{y}}. \tag{4.11}$$

For example, in the case of local $\mathbb{P}^2$, we simply obtain

$$\mathsf{O}_{\mathbb{P}^2} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{x}-\mathsf{y}}. \tag{4.12}$$

(A small comment on notation: when the toric CY is the canonical bundle of a complex surface S, we will denote O_S instead of O_X). Another frequently used example is local $\mathbb{F}_0$. The resulting operator in this case is

$$\mathsf{O}_{\mathbb{F}_0} = \mathrm{e}^{\mathsf{x}} + \xi \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}, \tag{4.13}$$

where ξ is a "mass parameter" of the CY, in principle complex. Let us point out that, In the case of mirror curves of genus g_Σ, there are g_Σ different operators associated to g_Σ different canonical forms of the curve, see [84] for a detailed explanation.

What kind of operators are the O_X? First of all, since the momentum operator is exponentiated, it acts as a translation operator on wavefunctions,

$$\mathrm{e}^{a\mathsf{y}} \psi(x) = \psi(x - a\mathrm{i}\hbar). \tag{4.14}$$

Therefore, the operators above are functional-difference operators. Next, we will regard these operators as acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Their domain consists of functions in $\mathcal{H}$ which can be extended to a strip $\mathbb{R} \times [-L, L]$ in the complex plane, and in such a way that they remain square integrable along the lines $\mathbb{R} + \mathrm{i}y$, $y \in [-L, L]$. The value of L depends on the operator.

Once we have defined these operators on a Hilbert space, we can ask all the usual questions that we ask in quantum mechanics. Are they actually self-adjoint? If yes, what are their spectral properties? It turns out that, provided some obvious conditions are imposed on the values of the parameters, the operators obtained from mirror curves are self-adjoint and have a *discrete* spectrum (in the case of the operator (4.13), the condition on the parameter is $\xi > 0$.) The discreteness of the spectrum was first shown numerically in [85], and then it was proved in [86–88] as a consequence of a stronger result. Namely, it was shown that the inverse operator

$$\rho_X = \mathsf{O}_X^{-1} \tag{4.15}$$

exists and is of trace class, i.e. it satisfies

$$\mathrm{Tr}\, \rho_X < \infty. \tag{4.16}$$

This implies that the spectrum of ρ_X is discrete, with an accumulation point at the origin. It follows that the spectrum of O_X is also discrete.

Exercise 4.1. Use numerical techniques to calculate the spectrum of (4.12) for various values of $\hbar$. Show in particular that, for $\hbar = 2\pi$, if we write the spectrum as e^{E_n}, $n = 0, 1, \cdots$, one finds, for the very first levels, the table 1. □

n	E_n
0	2.56264206862381937
1	3.91821318829983977
2	4.91178982376733606
3	5.73573703542155946
4	6.45535922844299896

Table 1. Numerical spectrum of the operator (4.12) for $n = 0, 1, \cdots, 4$, and $\hbar = 2\pi$.

Remark 4.2. Although we also use the expression "quantum curve," our approach is very different from the one used by practicioners of topological recursion and explained in the courses by V. Bouchard and K. Iwaki. In that approach, one has a formal (wave)function, written as a perturbative series in $\hbar$, and looks for a formal differential operator which annihilates it. In particular, there is no notion of Hilbert space. In our approach, in contrast, the operator is given from the very beginning by the standard Weyl quantization of the mirror curve, and it is defined on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. The spectrum and eigenfunctions are well-defined for $\hbar > 0$, and one then looks for the relation between these non-perturbative data and the geometric content of the topological string.

Trace class operators are in many ways the best possible operators in spectral theory. One can show [89] that, if an operator ρ_X is trace class, the traces of ρ_X^n are all finite, for $n \in \mathbb{Z}_{>0}$, and in addition that the *Fredholm or spectral determinant*

$$\Xi_X = \det\left(1 + \kappa\rho_X\right) \tag{4.17}$$

is an *entire* function of κ. The Fredholm determinant can be regarded as an infinite-dimensional generalization of the characteristic polynomial of a Hermitian matrix. Just as the zeroes of the characteristic polynomial give the eigenvalues of the matrix, the spectrum of the operator ρ_X can be read from the zeroes of the Fredholm determinant: if we denote by e^{-E_n} the eigenvalues of ρ_X, $n \in \mathbb{Z}_{\geq 0}$, the Fredholm determinant vanishes at

$$\kappa = -\mathrm{e}^{E_n}. \tag{4.18}$$

In addition, there is an infinite-dimensional version of the factorization of a characteristic polynomial, and the Fredholm determinant (4.17) has the infinite product representation [89]

$$\Xi_X(\kappa, \hbar) = \prod_{n=0}^{\infty}\left(1 + \kappa\mathrm{e}^{-E_n}\right). \tag{4.19}$$

It is easy to see that we can identify κ with the modulus of the CY appearing in (4.10).

Since the Fredholm determinant is an entire function, it has a convergent power series expansion around $\kappa = 0$, of the form

$$\Xi_X(\kappa, \hbar) = 1 + \sum_{N=1}^{\infty} Z_X(N, \hbar)\kappa^N. \tag{4.20}$$

We will refer to the coefficients $Z_S(N, \hbar)$ as *fermionic spectral traces*. They can be defined as [89]

$$Z_X(N, \hbar) = \mathrm{Tr}\left(\Lambda^N(\rho_X)\right), \qquad N = 1, 2, \cdots \tag{4.21}$$

In this expression, the operator $\Lambda^N(\rho_X)$ is defined by $\rho_X^{\otimes N}$ acting on $\Lambda^N\left(L^2(\mathbb{R})\right)$. They can be also obtained from the more conventional, "bosonic" traces $\mathrm{Tr}\rho_X^\ell$, since one has

$$\log \Xi_X(\kappa, \hbar) = -\sum_{\ell=1}^{\infty} \frac{(-\kappa)^\ell}{\ell} \mathrm{Tr}\rho_X^\ell. \tag{4.22}$$

Exercise 4.3. Let us consider the spectrum $\lambda_n = n^{-2}$, $n = 1, 2, \cdots$. This can be realized for example as minus the spectrum of the inverse Laplacian on the circle, after removing the zero mode. Show that the corresponding Fredholm determinant is given by [90]

$$\Xi(\kappa) = \prod_{n=1}^{\infty} \left(1 + \frac{\kappa}{n^2}\right) = \frac{\sinh\left(\pi\kappa^{1/2}\right)}{\pi\kappa^{1/2}}, \tag{4.23}$$

while the bosonic traces are $\zeta(2\ell)$, $\ell = 1, 2, \cdots$.

One of the main results of [11, 12, 84] is that the fermionic spectral traces $Z_X(N, \hbar)$ are non-perturbative completions of the total topological string free energy in the conifold frame. More precisely, we have the following

Conjecture 4.4. Let us consider the 't Hooft limit

$$N \to \infty, \quad \hbar \to \infty, \qquad \frac{N}{\hbar} \qquad \text{fixed}. \tag{4.24}$$

Then, $Z_X(N, \hbar)$ has conjecturally an asymptotic expansion of the form

$$\log\, Z_X(N, \hbar) \sim \sum_{g \geq 0} F_g^c(\lambda) g_s^{2g-2}. \tag{4.25}$$

The relation between the parameters in the two sides the following. The string coupling constant is related to $\hbar$ by

$$g_s = \frac{4\pi^2}{\hbar}. \tag{4.26}$$

$F_g^c(\lambda)$ is the genus g free energy of X in the conifold frame, and the 't Hooft parameter

$$\lambda = N g_s \tag{4.27}$$

is identified with the canonically normalized conifold coordinate.

Let us make some comments on this conjecture. First of all, we note that the limit (4.24) is indeed very similar to the 't Hooft limit (4.1) that we introduced in the case of the Gopakumar–Vafa duality. An interesting point is that, as shown in (4.26), the coupling constant is essentially the inverse of $\hbar$. Therefore, the weakly coupled limit of the topological string corresponds to the strong coupling limit of the quantum mechanical problem. When the quantization of mirror curves was initially proposed in [5], it was hoped that the topological string would emerge in the weak coupling limit, but it was later realized that this limit corresponds to the NS topological string mentioned in section 3.3. The emergence of the conventional topological string in the strong coupling limit was observed in a different line of work on localization and ABJM theory [91–94]. We should also note in this respect that, for the operators in exponentiated variables appearing in the TS/ST correspondence, there is a conjectural strong-weak coupling duality in

$\hbar$ [95], and the strong coupling limit of the spectrum $\hbar \to \infty$ can be related to the weak coupling limit $\hbar \to 0$. This is expected to be related to the modular duality for Weyl operators noted by Faddeev in [44].

The conjecture (4.25) also suggests that $Z_X(N, \hbar)$ plays the rôle of a partition function in a quantum field theory, where N is the rank. Although there is no explicit realization of such a quantum field theory for the moment being, in concrete examples one can relate the fermionic spectral traces to matrix integrals. Indeed, a theorem of Fredholm asserts that, if $\rho_X(p_i, p_j)$ is the kernel of ρ_X, the fermionic spectral trace can be computed as an N-dimensional integral,

$$Z_X(N, \hbar) = \frac{1}{N!} \int \det \left(\rho_X(p_i, p_j) \right) \, \mathrm{d}^N p. \tag{4.28}$$

In cases where the kernel ρ_X can be computed explicitly, the above integral can be written as an eigenvalue integral, and analyzed with matrix model techniques [12, 87, 96], as we will see in examples in the next section.

Remark 4.5. Physically, (4.28) can be interpreted as the canonical partition function of a non-interacting Fermi gas described by the one-body density matrix ρ_X. It is then natural to define the Hamiltonian H_X of such a system by $\rho_X = \mathrm{e}^{-\mathsf{H}_X}$. In this picture, the Fredholm determinant $\Xi_X(\kappa)$ is the grand-canonical partition function of the Fermi gas, and $\kappa = \mathrm{e}^{\mu}$ is the exponent of the fugacity in the grand-canonical ensemble. This is somewhat similar to the Fermi picture of matrix models discussed in C. Johnson's lectures.

The conjecture (4.25) is in fact a consequence of a stronger conjecture formulated in [11]. This conjecture gives an *exact* expression for $Z_X(N, \hbar)$ and for the Fredholm determinant $\Xi_X(\kappa)$ in terms of the GV free energy and additional enumerative information, see e.g. [97] for a review.

4.3 Local $\mathbb{P}^2$, non-perturbatively

The conjecture (4.25) seems difficult to prove, since it relates a quantum mechanical model at strong coupling (i.e. for $\hbar \to \infty$) to topological string theory on a toric CY theefold. It is however a falsifiable statement, i.e. we can calculate both sides of the conjecture and check whether they are equal or not. So far all tests have been successful. Many of these tests involve the stronger form of the conjecture mentioned above, and they are typically numerical, since it is easier to calculate the fermionic spectral traces numerically for low values of N, than to compute their asymptotic behavior in the 't Hooft limit. In some cases, however, it is also possible to calculate this asymptotic expansion explicitly. We will now consider the spectral theory associated to local $\mathbb{P}^2$, where many explicit results can be obtained.

The starting point of this analysis is the fact that, for some mirror curves, the integral kernel of the operator ρ_X can be explicitly and exactly computed.

Remark 4.6. This is remarkable since I don't know of any other trace class operator in one-dimensional quantum mechanics where this can be done. For example, the monic potentials

$$\mathsf{H}_\ell = \mathsf{y}^2 + \mathsf{x}^{2\ell}, \qquad \ell \geq 2 \tag{4.29}$$

have been extensively studied and their inverses are of trace class, but no exact expression is known for the integral kernel of H_ℓ^{-1}.

Let us consider the following family of operators:

$$\mathsf{O}_{m,n} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-m\mathsf{x} - n\mathsf{y}}, \quad m, n \in \mathbb{R}_{>0}. \tag{4.30}$$

They were called three-term operators in [86]. Note that the case $m = n = 1$ corresponds to local $\mathbb{P}^2$ (the case with arbitrary positive integers m, n corresponds to the toric CY given by the canonical bundle on the weighted projective space $\mathbb{P}(1, m, n)$). We now define the function

$$\Psi_{a,c}(x) = \frac{\mathrm{e}^{2\pi a x}}{\Phi_{\mathsf{b}}(x - \mathsf{i}(a + c))}, \tag{4.31}$$

involving Faddeev's quantum dilogarithm, where a, c are positive real numbers. Let us now introduce normalized Heisenberg operators q, p, satisfying the normalized commutation relation

$$[\mathsf{p}, \mathsf{q}] = (2\pi \mathsf{i})^{-1}. \tag{4.32}$$

They are related to x, y by the linear canonical transformation,

$$\mathsf{x} = 2\pi \mathsf{b} \frac{(n+1)\mathsf{p} + n\mathsf{q}}{m+n+1}, \quad \mathsf{y} = -2\pi \mathsf{b} \frac{m\mathsf{p} + (m+1)\mathsf{q}}{m+n+1}. \tag{4.33}$$

In particular, $\hbar$ is related to b by

$$\hbar = \frac{2\pi \mathsf{b}^2}{m+n+1}. \tag{4.34}$$

It was proved in [86] that, in the momentum representation associated to p, the integral kernel of the operator $\rho_{m,n}$ can be written explicitly in terms of the function (4.31). It reads,

$$\rho_{m,n}(p, p') = \frac{\overline{\Psi_{a,c}(p)}\, \Psi_{a,c}(p')}{2\mathsf{b} \cosh\left(\pi \frac{p - p' + \mathsf{i}(a + c - nc)}{\mathsf{b}}\right)}. \tag{4.35}$$

In this equation, a, c are given by

$$a = \frac{m\mathsf{b}}{2(m+n+1)}, \qquad c = \frac{\mathsf{b}}{2(m+n+1)}. \tag{4.36}$$

As we will see in the following exercises, this results allows for many explicit calculations.

Exercise 4.7. In this exercise you are asked to use the explicit formula (4.35) to calculate the first trace of $\rho_{m,n}$, given by

$$\mathrm{Tr}\rho_{m,n} = \int_{\mathbb{R}} \rho_{m,n}(p, p) \mathrm{d}p. \tag{4.37}$$

First note that, by using the property (B.8), one can write

$$|\Psi_{a,c}(p)|^2 = \mathrm{e}^{4\pi a p} \frac{\Phi_{\mathsf{b}}(p + \mathsf{i}(a + c))}{\Phi_{\mathsf{b}}(p - \mathsf{i}(a + c))}, \tag{4.38}$$

therefore

$$\rho_{m,n}(p, p) = \frac{\Phi_{\mathsf{b}}(p + \mathsf{i}(a + c))}{\Phi_{\mathsf{b}}(p - \mathsf{i}(a + c))} \frac{\mathrm{e}^{4\pi a p}}{2\mathsf{b} \cos\left(\pi \frac{a + c - nc}{\mathsf{b}}\right)}. \tag{4.39}$$

The first trace can be computed explicitly for arbitrary values of m, n and $\hbar$, by using properties of Faddeev's quantum dilogarithm, as shown in [86]. For this exercise we will consider the case

$$\hbar = 2\pi. \tag{4.40}$$

This is a special value of $\hbar$ where the theory of quantum mirror curves simplifies very much, as shown in [11]. We will also take $n = 1$ and m an arbitrary positive integer. For these values we have

$$\mathsf{b}^2 = m + 2, \tag{4.41}$$

and

$$a + c = \frac{1}{2}\left(\mathsf{b} - \mathsf{b}^{-1}\right). \tag{4.42}$$

Show, by using the properties (B.9a), (B.9b), that

$$\frac{\Phi_{\mathsf{b}}\left(p + \frac{\mathrm{i}}{2}\left(\mathsf{b} - \mathsf{b}^{-1}\right)\right)}{\Phi_{\mathsf{b}}\left(p - \frac{\mathrm{i}}{2}\left(\mathsf{b} - \mathsf{b}^{-1}\right)\right)} = \frac{1 - \mathrm{e}^{2\pi \mathsf{b}^{-1} p}}{1 - \mathrm{e}^{2\pi \mathsf{b} p}}. \tag{4.43}$$

Deduce the following expression:

$$\mathrm{Tr}\rho_{m,1} = \frac{1}{2\pi \cos\left(\frac{\pi m}{2(m+2)}\right)} \int_{\mathbb{R}} \mathrm{e}^{(m-1)y} \frac{\sinh(y)}{\sinh((m+2)y)} \mathrm{d}y, \tag{4.44}$$

where $y = \pi p/\mathsf{b}$. The integral can be evaluated e.g. by residues, and one concludes that [86]

$$\mathrm{Tr}\rho_{m,1}\left(\hbar = 2\pi\right) = \frac{1}{4(m+2)\sin\left(\frac{\pi}{m+2}\right)\sin\left(\frac{2\pi}{m+2}\right)}. \tag{4.45}$$

In particular, for $m = 1$, which corresponds to local $\mathbb{P}^2$, one finds

$$\mathrm{Tr}\rho_{1,1}\left(\hbar = 2\pi\right) = \frac{1}{9}. \tag{4.46}$$

□

Exercise 4.8. C. Johnson explained in his lectures how to compute Fredholm determinants numerically with the approach of [98]. That method requires an explicit knowledge of the kernel, which for three-term operators is given by (4.35). In the previous exercise we showed that the diagonal kernel $\rho_{m,1}(p,p)$ simplifies for $\hbar = 2\pi$. We can in fact simplify the whole kernel by using a similarity transformation [99],

$$\rho_X(p, p') \to h(p)\rho_X(p, p')(h(p'))^{-1}. \tag{4.47}$$

Such a transformation does not change the value of the traces of ρ_X^n, nor the spectral determinant. In the case of $\rho_{m,n}(p, p')$, we take

$$h(p) = \sqrt{\frac{\Psi_{a,c}(p)}{\overline{\Psi_{a,c}(p)}}}. \tag{4.48}$$

Show that, for $\hbar = 2\pi$, and after the similarity transformation, we can write

$$\rho_{1,1}(y, y') = \frac{1}{2\pi}\sqrt{\frac{\sinh(y)}{\sinh(3y)}} \frac{1}{\cosh\left(y - y' + \frac{\mathrm{i}\pi}{6}\right)} \sqrt{\frac{\sinh(y')}{\sinh(3y')}} \tag{4.49}$$

in terms of the variable y appearing in (4.44). Implement the algorithm of [98] to calculate $\Xi_{\mathbb{P}^2}(\kappa)$ numerically, and use this result to obtain the very first energy levels. As an example of what

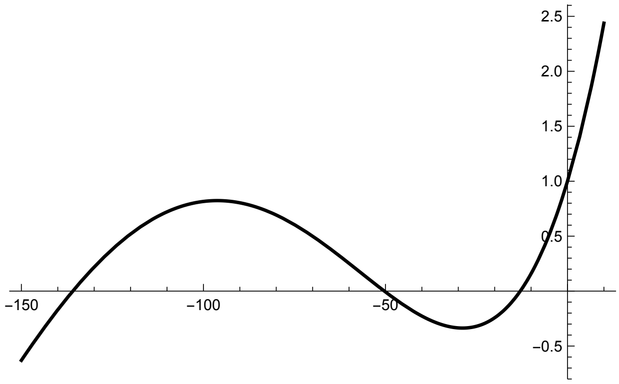

Figure 5. A numerical calculation of the Fredholm determinant $\Xi_{\mathbb{P}^2}(\kappa)$ for local $\mathbb{P}^2$.

you should find, in Fig. 5 I plot a numerical calculation of $\Xi_{\mathbb{P}^2}(\kappa)$, obtained as follows. First, I use a Gauss–Kronrod quadrature or order 50 to calculate the Fredholm determinant. I do this calculation for 150 values of κ in the interval $[-150, 10]$, and I construct an interpolating function. The zeroes of this interpolating function are approximately at -12.97004, -50.3105 and -135.882 (rounded numerical values), in good agreement with the result in table Fig. 1. Note that the plot in Fig. 5 is indistinguishable from the plot of the same quantity that appears in [97]. The latter was obtained by using the conjecture of [11], which expresses the Fredholm determinant as a "quantum theta function." The fact that they agree so precisely (within numerical errors) is an explicit test of the TS/ST correspondence. □

By using the explicit expression for the integral kernel of $\rho_{m,n}$ one can write down an explicit expression for the integral (4.28), which we will denote in this case as $Z_{m,n}(N, \hbar)$. As in [92, 100], we use Cauchy's identity:

$$\begin{aligned}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]}{\prod_{i,j}2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)} &= \det_{ij}\frac{1}{2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)} \\ &= \sum_{\sigma\in S_N}(-1)^{\epsilon(\sigma)}\prod_i\frac{1}{2\cosh\left(\frac{\mu_i-\nu_{\sigma(i)}}{2}\right)}.\end{aligned} \tag{4.50}$$

One finds then,

$$Z_{m,n}(N,\hbar) = \frac{1}{N!}\int_{\mathbb{R}^N}\frac{\mathrm{d}^N p}{\mathsf{b}^N}\prod_{i=1}^N|\Psi_{a,c}(p_i)|^2\,\frac{\prod_{i<j}4\sinh\left(\frac{\pi}{\mathsf{b}}(p_i-p_j)\right)^2}{\prod_{i,j}2\cosh\left(\frac{\pi}{\mathsf{b}}(p_i-p_j)+\mathrm{i}\pi C_{m,n}\right)}, \tag{4.51}$$

where

$$C_{m,n} = \frac{m-n+1}{2(m+n+1)}. \tag{4.52}$$

Remark 4.9. The above integral is real, since the kernel (4.35) is Hermitian. More importantly, it is also absolutely convergent for $\hbar > 0$, due to the trace class property. Therefore, this property guarantees that the fermionic spectral trace, and its matrix model realization, provides a well-defined non-perturbative completion of the topological string partition function. This is in contrast to doubly-scaled matrix models of two-dimensional gravity, which are often ill-defined non-perturbatively, at least with the standard choice of integration contours. In the

TS/ST correspondence, instabilities seem to appear only for some special ranges of the mass parameters, and they can be easily understood by using non–Hermitian extensions of quantum mechanics (see e.g. [101]).

In order to test the conjecture (4.25) we have to study the matrix integral (4.51) in the 't Hooft limit (4.24), therefore we should understand what happens to the integrand of (4.51) when $\hbar$ (or equivalently b) is large. To do this, we first change variables to

$$u_i = \frac{2\pi}{\mathsf{b}} p_i, \tag{4.53}$$

and we introduce the parameter

$$\mathsf{g} = \frac{1}{\hbar} = \frac{m+n+1}{2\pi} \frac{1}{\mathsf{b}^2}, \tag{4.54}$$

so that the the weak coupling regime of g is the strong coupling regime of $\hbar$. In general quantum-mechanical models, this regime is difficult to understand, but in this case we can use the crucial property of self-duality of Faddeev's quantum dilogarithm,

$$\Phi_{\mathsf{b}}(x) = \Phi_{1/\mathsf{b}}(x). \tag{4.55}$$

Then, by using (4.38), we can write

$$|\Psi_{a,c}(p)|^2 = \exp\left(\frac{mu}{2\pi\mathsf{g}}\right) \frac{\Phi_{1/\mathsf{b}}\left(\left(u + 2\pi\mathsf{i}(a+c)/\mathsf{b}\right)/2\pi\mathsf{b}^{-1}\right)}{\Phi_{1/\mathsf{b}}\left(\left(u - 2\pi\mathsf{i}(a+c)/\mathsf{b}\right)/2\pi\mathsf{b}^{-1}\right)}, \tag{4.56}$$

where u and p are related through (4.53). When b is large, $1/\mathsf{b}$ is small and we can use the asymptotic expansion (B.10). We define the *potential* of the matrix model as,

$$V_{m,n}(u,\mathsf{g}) = -\mathsf{g} \log |\Psi_{a,c}(p)|^2, \tag{4.57}$$

where u and p are related as in (4.53). By using (B.10), we deduce that this potential has an asymptotic expansion at small g, of the form

$$V_{m,n}(u,\mathsf{g}) = \sum_{\ell \geq 0} \mathsf{g}^{2\ell} V_{m,n}^{(\ell)}(u). \tag{4.58}$$

The leading contribution as $\mathsf{g} \to 0$ is given by the "classical" potential,

$$V_{m,n}^{(0)}(u) = -\frac{m}{2\pi} u - \frac{m+n+1}{2\pi^2} \mathrm{Im}\left(\mathrm{Li}_2\left(-\mathrm{e}^{u + \pi\mathrm{i}\frac{m+1}{m+n+1}}\right)\right). \tag{4.59}$$

Exercise 4.10. By using the asymptotics of the dilogarithm,

$$\mathrm{Li}_2(-\mathrm{e}^x) \approx \begin{cases} -x^2/2, & x \to \infty, \\ -\mathrm{e}^x, & x \to -\infty, \end{cases} \tag{4.60}$$

show that

$$V_{m,n}^{(0)}(u) \approx \begin{cases} \frac{u}{2\pi}, & u \to \infty, \\ -\frac{m}{2\pi} u, & u \to -\infty, \end{cases} \tag{4.61}$$

i.e. it is a linearly confining potential. This is similar to the potentials appearing in matrix models for Chern–Simons–matter theories (see e.g. [92]). □

We can now write the matrix integral as

$$Z_{m,n}(N,\hbar) = \frac{1}{N!}\int_{\mathbb{R}^N} \frac{\mathrm{d}^N u}{(2\pi)^N} \prod_{i=1}^{N} \mathrm{e}^{-\frac{1}{\mathsf{g}}V_{m,n}(u_i,\mathsf{g})} \frac{\prod_{i<j} 4\sinh\left(\frac{u_i-u_j}{2}\right)^2}{\prod_{i,j} 2\cosh\left(\frac{u_i-u_j}{2} + \mathrm{i}\pi C_{m,n}\right)}. \tag{4.62}$$

This expression is very similar to matrix models that have been studied before in the literature. The interaction between eigenvalues is similar to the matrix model (4.7) which appears in the Gopakumar–Vafa duality, and is identical to the one appearing in the generalized $O(2)$ models of [102], and in some matrix models for Chern–Simons–matter theories studied in for example [100]. The parameter g corresponds to the string coupling constant, but in contrast to conventional matrix models, the potential depends itself on g.

The above expression is perfectly suited to study the 't Hooft limit. One can use e.g. saddle-point techniques to solve for the leading behavior in the $1/N$ expansion or planar limit. This limit is described by the so-called planar resolvent and density of eigenvalues, and from this one can compute the genus zero free energy $F_0^c(\lambda)$. In this limit, only the classical part of the potential (4.59) has to be taken into account. The resolvent for the local $\mathbb{P}^2$ matrix model, given by (4.62) with $m = n = 1$) was first conjectured in [103]. This conjecture was later proved in a *tour de force* calculation in [96], by extending the techniques of [102]. Let us introduce the exponentiated variable

$$X = \mathrm{e}^u \tag{4.63}$$

where u is the variable appearing in (4.62). Then, the spectral curve describing the planar limit of the matrix model is given by

$$X^3 + Y^{-3} + \kappa X Y^{-1} + 1 = 0, \tag{4.64}$$

which can be easily seen to be a reparametrization of the classical mirror curve (2.45). As it is standard in the study of the planar limit, the density of eigenvalues can be written as the discontinuity of the so-called planar resolvent [104], see [78, 105] for a review (this was also mentioned in C. Johnson's lectures)

$$\rho(u) = \frac{1}{2\pi\mathrm{i}}\left(\omega(X - \mathrm{i}0) - \omega(X + \mathrm{i}0)\right), \tag{4.65}$$

where in this case

$$\omega(X) = \frac{3\mathrm{i}}{\pi}\frac{\log Y(X)}{X}. \tag{4.66}$$

(The actual planar resolvent has an additional piece, but it does not contribute to the density of eigenvalues). The density $\rho(u)$ is a symmetric one-cut distribution, and the end-points of the cut can be found from the branch cuts of the spectral curve. They are given by $\pm a$, where

$$a = -\frac{1}{3}\log\left(-\frac{2}{27}\kappa^3 - 1 - \frac{2}{27}\sqrt{\kappa^6 + 27\kappa^3}\right), \tag{4.67}$$

and we are assuming that $-\infty < \kappa < -3$ which corresponds to the region $-1/27 < z < 0$. One way of testing the result above for $\rho(u)$ is to consider a finite but large number N of "typical eigenvalues" and see how they distribute along a histogram. This distribution should approximate the density of eigenvalues $\rho(u)$ as N grows large. In the lectures of C. Johnson you obtained such a distribution in the case of the Gaussian matrix model by generating a random matrix with

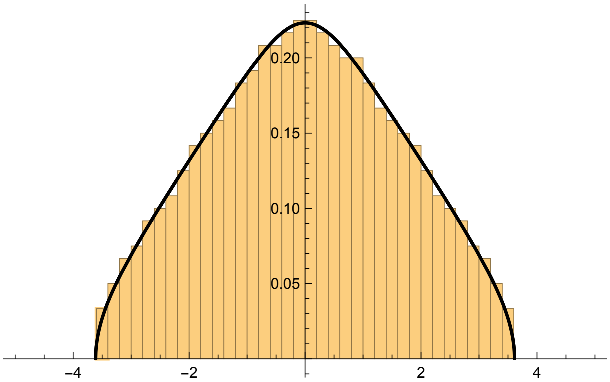

Figure 6. This figure shows a histogram of the equilibrium eigenvalues of the local $\mathbb{P}^2$ matrix model, for $N = 600$ and $\kappa = -70$, together with the density of eigenvalues $\rho(u)$ (the black line).

Gaussian weight, and then calculating their eigenvalues. Here the probability distribution is not Gaussian, and we will obtain the distribution in a different way, as follows. Let us consider an eigenvalue integral of the form

$$Z(N) = \int \prod_{i=1}^{N} \mathrm{e}^{-\sum_{i=1}^{N} v(x_i)} \prod_{i<j} \mathcal{I}(x_i - x_j). \tag{4.68}$$

The saddle point at finite N is the configuration $x_1, \cdots, x_N$ that minimizes the "effective action"

$$S(x_1, \cdots, x_N) = \sum_{i=1}^{N} v(x_i) - \sum_{i<j} \log \mathcal{I}(x_i - x_j). \tag{4.69}$$

This action can be regarded as a generalization of the Dyson gas. In the limit of large N the minimization is described by the eigenvalue distribution $\rho(x)$, but the minimization problem can be solved for any finite N, under appropriate conditions. We will call the $x_1, \cdots, x_N$ minimizing (4.69) the *equilibrium eigenvalues*[1]. To find these eigenvalues in the case of (4.62) we use only the classical potential (4.59). In Fig. 6 we show both, the histogram for the equilibrium eigenvalues obtained from the matrix model for local $\mathbb{P}^2$ with $N = 600$ and $\kappa = -70$, and the density of eigenvalues $\rho(u)$ obtained from (4.65), (4.66). The latter provides an excellent approximation to the former.

The calculation of the subleading terms of the $1/N$ expansion is complicated. Ideally, one would like to show that the matrix integral (4.62) satisfies topological recursion, and together with the remodeling conjecture of [9] (which is now a theorem), one would have a proof of the conjecture (4.25) for local $\mathbb{P}^2$ and some other cases. This has not been achieved so far. However, it is still possible to test the conjecture (4.25) by calculating the matrix integral in a perturbative expansion in g, at fixed N. This can be used to obtain the expansion of $F_g^c(\lambda)$ around $\lambda = 0$, as shown in detail [12], and it can be verified that the result agrees with (2.61), (2.62).

[1]This procedure is different from the one discussed in C. Johnson's lectures. He looks at realizations of a random Gaussian distribution at finite N, and different realizations lead to different lists of eigenvalues. Since the large N limit implements at the same time a classical and a thermodynamic limit, his calculation takes into account both quantum and finite size effects at finite N. In my calculation I consider already the classical limit of the problem, described by the generalized Dyson gas (4.69), and therefore by working at finite N I take into account only finite size effects. Of course, as N becomes large, both calculations converge to the same probability distribution.

The discussion above makes it clear that the conjecture (4.25) can be regarded as an explicit realization of "quantum geometry," in which the spacetime geometry of the toric CY, together with its "stringy" corrections (given by embedded Rieman surfaces) *emerges* from a simple quantum mechanical model on the real line. As in other string/gauge theory dualities of the 't Hooft type, in the quantum model the CY modulus λ is "quantized" in units of the string coupling constant. We can also think about the geometry of the CY as emerging from the eigenvalue integral (4.28) in the 't Hooft limit. This limit is encoded in a spectral curve, which is nothing but the mirror curve we started with. Here we have seen how this works in the explicit example of local $\mathbb{P}^2$, and one can also work out the case of local $\mathbb{F}_0$ [96, 103]. We expect however this picture to hold for general toric manifolds, namely the 't Hooft limit of the spectral traces should be described by a spectral curve given by the mirror curve. Higher order corrections to the spectral trace in the $1/N$ expansion should be governed by the topological recursion.

We have provided a non-perturbative definition of topological string theory on toric CY manifolds in terms of simple quantum-mechanical models. The reader could ask why is this definition special. Certainly, non-perturbative definitions are not unique, and the TS/ST correspondence has not been justified so far as a full-fledged string/gauge theory duality. One can argue however that this non-perturbative definition is mathematically very rich and non-trivial. It leads to remarkable conjectures on the exact spectrum of quantum mirror curves, and by using the formulation of [95] it also leads to exact quantization conditions [106, 107] for cluster integrable systems associated to toric CY manifolds [7]. It makes remarkable predictions on the classical problem of evaluating CY periods at the conifold point, and these predictions have been verified in many cases with sophisticated tools in algebraic geometry [108–110]. Finally, there is a very interesting limit of the conjecture (4.25) which makes contact with $\mathcal{N} = 2$ gauge theories in four dimensions [111–113]. For example, one can take as a starting point (4.25) as applied to local $\mathbb{F}_0$ and deduce from there that the matrix model

$$Z(N; g_s) = \frac{1}{N!} \int \frac{\mathrm{d}^N x}{(2\pi)^N} \prod_{i=1}^{N} \mathrm{e}^{-\frac{4}{g_s}\cosh(x_i)} \prod_{i<j} \tanh^2\left(\frac{x_i - x_j}{2}\right) \tag{4.70}$$

has an asymptotic expansion

$$\log Z(N; g_s) \sim \sum_{g \geq 0} \mathcal{F}_g(t) g_s^{2g-2}, \tag{4.71}$$

where $\mathcal{F}_g(t)$ are the SW free energies in the magnetic frame discussed in section 3.3. This last statement can be rigorously proved [111, 113], providing yet additional evidence for the TS/ST correspondence.

As a final comment, one should ask what is the relation between the contents of this section and the resurgent story explained in section 3. Resurgence suggests that the asymptotic expansion (4.25) can be promoted to an exact formula for the fermionic spectral traces, by using (lateral) Borel resummation and including perhaps trans-series. This formula should read,

$$\log Z_X(N, \hbar) = s_\pm \left(F_X^c\right)(N, \hbar) + \cdots \tag{4.72}$$

where F^c is the perturbative topological string series, and the dots in the r.h.s. represent possible additional trans-series. Optimistically, these trans-series are the ones appearing in the resurgent structure unveiled in section 3. This issue was addressed in [114] in the case of local $\mathbb{P}^2$. That work established without any reasonable doubt that the exact fermionic spectral traces are *different* from the Borel resummation of just the perturbative series. For example, when $N = 1$ and $\hbar = 2\pi$, the Borel resummation gives

$$s_{\pm}\left(F^c\right)(1, 2\pi) = -2.197217... \tag{4.73}$$

while the exact result was obtained in (4.46),

$$\log Z_{\mathbb{P}^2}(1, 2\pi) = -\log(9) = -2.197224... \tag{4.74}$$

Therefore, additional trans-series are clearly needed. Some numerical evidence was given in [114] that the trans-series appearing in the resurgent structure can provide the required corrections, but more work should be done in order to understand the non-perturbative effects implicit in the TS/ST correspondence.

Acknowledgements

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A A short review of resurgence

In this section we will review some of the results in resurgence which we will need. I will be brief since I. Aniceto has covered the topic in this school. A wonderful introduction to the subject can be found in [115]. More formal developments can be studied in [116, 117]. A more physical perspective can be found in [105, 118, 119].

Let

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \tag{A.1}$$

be a factorially divergent, formal power series in z, i.e. $a_n \sim n!$ (such series are also called Gevrey-1). Its *Borel transform* is given by

$$\widehat{\varphi}(\zeta) = \sum_{n\geq 0} a_n \frac{\zeta^n}{n!}. \tag{A.2}$$

Remark A.1. Sometimes it is more convenient to use a different version of the Borel transform (used e.g. in [105, 117]), and given by

$$\widetilde{\varphi}(\zeta) = \sum_{n\geq 1} a_n \frac{\zeta^{n-1}}{(n-1)!}. \tag{A.3}$$

It is related to the previous definition by

$$\widetilde{\varphi}(\zeta) = \frac{\mathrm{d}\widehat{\varphi}(\zeta)}{\mathrm{d}\zeta}. \tag{A.4}$$

We recall that a resurgent function is a Gevrey-1 series $\varphi(z)$ whose Borel transform has the following property: on any line issuing from the origin, there is a finite set of points (the singularities of the Borel transform) such that $\widehat{\varphi}(\zeta)$ may be continued analytically along any path that follows the line, while circumventing (from above or from below) those singular points. A resurgent function is *simple* if the singularities of its Borel transform are simple poles or logarithmic branch cuts.

In the following we will assume that our series are resurgent functions. In the case of the topological string, everything indicates that the only singularities are logarithmic we therefore in the following we will mostly focus on simple resurgent functions.

If $\widehat{\varphi}(\zeta)$ is simple and it has a singularity at $\zeta = \zeta_\omega$, its local expansion there is of the form

$$\widehat{\varphi}(\zeta_\omega + \xi) = -\frac{\mathsf{S}}{2\pi} \left\{ \frac{a}{\xi} + \log(\xi) \sum_{n \geq 0} \widehat{c}_n \xi^n \right\} + \text{regular}, \tag{A.5}$$

where the series

$$\sum_{n \geq 0} \widehat{c}_n \xi^n \tag{A.6}$$

has a finite radius of convergence. We note that we might want to make specific choices of normalization for the coefficients a, c_n, and that's why we have introduced an additional (in general complex) number S in (A.5), which is called a *Stokes constant*. We can now associate to the expansion around the singularity (A.5) the following factorially divergent series

$$\varphi_\omega(z) = \frac{a}{z} + \sum_{n \geq 0} c_n z^n, \qquad c_n = n! \widehat{c}_n. \tag{A.7}$$

Therefore, given a formal power series $\varphi(z)$, the expansion of its Borel transform around its singularities generates additional formal power series:

$$\varphi(z) \to \{\varphi_\omega(z)\}_{\omega \in \Omega}, \tag{A.8}$$

where Ω labels the set of singular points. We will call the set of functions $\varphi_\omega(z)$, together with their Stokes constants S_ω, the *resurgent structure* associated to the original series $\varphi(z)$.

A basic result of resurgence is that the new series $\varphi_\omega(z)$ "resurge" in the original series through the behavior of the coefficients a_k when k is large. Let $\varphi(z)$ be a simple resurgent function, and Let A be the singularity of the Borel transform which is closest to the origin in the complex ζ plane (we will assume for simplicity that there is only one, although the generalization is straightforward). Let the behavior near this singularity be as in (A.5), with $\zeta_\omega = A$. Then, the coefficients a_k have the following asymptotic behavior,

$$a_k \sim \frac{\mathsf{S}a}{2\pi} A^{-k-1} \Gamma(k+1) + \frac{\mathsf{S}}{2\pi} \sum_{n \geq 0} A^{-k+n} c_n \Gamma(k-n), \qquad k \gg 1. \tag{A.9}$$

To understand this formula better, it is convenient to write explicitly the very first terms:

$$a_k \sim \frac{\mathsf{S}a}{2\pi} A^{-k-1} \Gamma(k+1) + \frac{\mathsf{S}}{2\pi} A^{-k} \Gamma(k) \left\{ c_0 + \frac{c_1 A}{k-1} + \frac{c_2 A^2}{(k-1)(k-2)} + \cdots \right\}, \qquad k \gg 1. \tag{A.10}$$

The first factor in the r.h.s. gives the leading factorial asymptotics, while the second factor gives a series of corrections in $1/k$ to the leading factorial behavior. These corrections involve the coefficients a, c_n of the power series obtained in (A.7). One can use this asymptotic formula in two ways: as a procedure to extract the numbers A, a, c_n, S from the knowledge of the series a_k, of conversely, as a way to obtain the large order asymptotics of these coefficients once these numbers are known.

A very convenient way to encode the information in the singularities is through the notion of Stokes automorphism. Let us start with some definitions.

Let ζ_ω be a singularity of $\widehat{\varphi}(\zeta)$. A ray in the Borel plane which starts at the origin and passes through ζ_ω is called a *Stokes ray*. It is of the form $\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$, where $\theta = \arg(\zeta_\omega)$. Note that a Stokes ray might pass through many singularities. A typical situation is that we have a ray of singularities of the form $\ell\mathcal{A}$, where $\ell \in \mathbb{Z}_{>0}$.

Let $\varphi(z)$ a Gevrey-1 formal power series series, $z \in \mathbb{C}$, and $\theta = \arg z$. If $\widehat{\varphi}(\zeta)$ analytically continues to an L^1-analytic function along the ray $\mathcal{C}^\theta := \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$ we define its Laplace transform by

$$s(\varphi)(z) = \int_0^\infty \widehat{\varphi}(z\zeta)\mathrm{e}^{-\zeta}\mathrm{d}\zeta = \frac{1}{z}\int_{\mathcal{C}^\theta} \widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{A.11}$$

The function $s(\varphi)(z)$ is often called the *Borel resummation* of the formal power series $\varphi(z)$.

Let us first note that, if $s(\varphi)(z)$ exists, its asymptotic behavior for small z can be obtained by expanding the integrand and integrating term by term:

$$s(\varphi)(z) \sim \sum_{n\geq 0} a_n z^n. \tag{A.12}$$

This is the formal power series that we started with. Therefore, if we are lucky, Borel resummation produces an actual function which reproduces the original series. It is then a way to "make sense" of our original formal power series.

If we vary $\theta = \arg z$ and we do not encounter singularities of $\widehat{\varphi}$, the function $s_\theta(\varphi)(z)$ is locally analytic. However, the Borel transform is not in principle well defined when z lies on a Stokes ray, i.e. when $\theta = \arg(z)$ is the argument of a singularity of the Borel transform. In fact, as z crosses a Stokes ray, the Borel resummation has a discontinuity. To define this discontinuity more precisely, we introduce *lateral Borel resummations*.

Let $\varphi(z)$ be a resurgent function, and let $\mathcal{C}^\theta_\pm$ be contours starting at the origin and going slightly above (respectively, below) the Stokes ray, in such a way that $\mathcal{C}^\theta_+ - \mathcal{C}^\theta_-$ is a clockwise contour. Then, the lateral Borel resummations of $\varphi(z)$ are defined as

$$s_\pm(\varphi)(z) = \frac{1}{z}\int_{\mathcal{C}^\theta_\pm} \widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{A.13}$$

The discontinuity is then defined by

$$\mathrm{disc}(\varphi)(z) = s_+(\varphi)(z) - s_-(\varphi)(z). \tag{A.14}$$

Note that, since $s_\pm(\varphi)(z)$ have the same asymptotics for small z, given in (A.12), the discontinuity must be invisible as an asymptotic expansion. As we will now see, this difference is *exponentially small* and closely related to the local structure of the Borel transform. Indeed, let us assume that $\varphi(z)$ is a simple resurgent function, with a sequence of isolated logarithmic singularities ζ_ω in the Stokes ray, where $\omega \in \Omega$. The difference between the two contours $\mathcal{C}^\theta_+ - \mathcal{C}^\theta_-$ can be deformed into a sum of Hankel-like contours γ_ω around the logarithmic branch cuts. We then have, for each ω,

$$\oint_{\gamma_\omega} \widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta = -\frac{\mathrm{e}^{-\zeta_\omega/z}}{2\pi}\int_{\mathcal{C}^\theta_-} (\log(\xi) - \log(\xi) - 2\pi\mathrm{i})\,\widehat{\varphi}_\omega(\xi)\mathrm{e}^{-\xi/z}\mathrm{d}\xi, \tag{A.15}$$

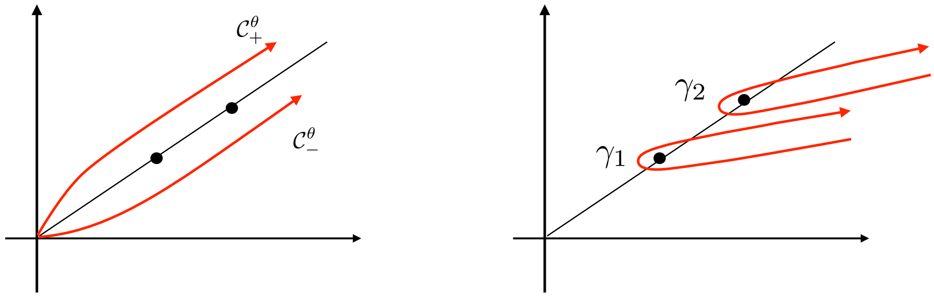

Figure 7. Contour deformation in the calculation of the discontinuity.

where in the first line we have written $\zeta = \zeta_\omega + \xi$. Therefore

$$\begin{aligned} s_+(\varphi)(z) - s_-(\varphi)(z) &= \frac{1}{z}\sum_{\omega\in\Omega}\oint_{\gamma_\omega}\widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta = \mathrm{i}\sum_{\omega\in\Omega}\frac{\mathrm{e}^{-\zeta_\omega/z}}{z}\int_{\mathcal{C}_-^\theta}\widehat{\varphi}_\omega(\xi)\mathrm{e}^{-\xi/z}\mathrm{d}\xi \\ &= \mathrm{i}\sum_{\omega\in\Omega}\mathrm{e}^{-\zeta_\omega/z}s_-(\varphi_\omega)(z). \end{aligned} \tag{A.16}$$

If the simple resurgent function has in addition pole singularities, the pole in (A.5) gives a contribution

$$\mathrm{i}\,\frac{a_\omega}{z} \tag{A.17}$$

to the discontinuity, and the formula (A.16) still holds with the definition (A.7).

The expression (A.16) involves (possible infinite) sums of the series $\varphi_\omega(z)$ with an exponentially small prefactor $\mathrm{e}^{-\zeta_\omega/z}$. These objects are called *trans-series*. More formally, let $\varphi_\omega(z)$ be resurgent functions. A *trans-series* is a (possibly infinite) formal linear combination of formal power series

$$\Phi(z;\boldsymbol{C}) = \sum_\omega C_\omega \mathrm{e}^{-\zeta_\omega/z}\varphi_\omega(z), \tag{A.18}$$

where $\boldsymbol{C} = (C_{\omega_1}, \cdots)$ is a (possibly infinite) vector of complex numbers.

The result (A.16) involves Borel resummed trans-series, but it is useful to rewrite it as a relation between formal trans-series themselves. If we regard lateral Borel resummations as operators, we introduce the *Stokes automorphism* along the ray $\mathcal{C}^\theta$, $\mathfrak{S}_\theta$, as

$$s_+ = s_-\mathfrak{S}_\theta. \tag{A.19}$$

Then, we can write (A.16) as

$$\mathfrak{S}_\theta(\varphi) = \varphi + \mathrm{i}\sum_{\omega\in\Omega}\mathsf{S}_\omega \mathrm{e}^{-\zeta_\omega/z}\varphi_\omega(z). \tag{A.20}$$

Let us note that, when we use the definition of Borel transform in (A.3), the Borel resummation is defined by the formula

$$s_\pm(\varphi)(z) = a_0 + \int_{\mathcal{C}_\pm^\theta}\widetilde{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{A.21}$$

This can be used to calculate the discontinuity or Stokes automorphism through a Stokes ray, and the resulting trans-series will be independent on whether we used the first definition of Borel transform (A.2) or the second (A.3).

We will now state a principle of *semiclassical decoding*.

Definition A.2. (Semiclassical decoding). Let $f(z)$ be a function with the asymptotic expansion

$$f(z) \sim \varphi(z) = \sum_{n \geq 0} a_n z^n. \tag{A.22}$$

We say that $f(z)$ admits a *semiclassical decoding* if $\varphi(z)$ can be promoted to a trans-series $\Phi(z; \boldsymbol{C})$, which is lateral Borel summable, and such that

$$f(z) = s_{\pm}(\Phi)(z; \boldsymbol{C}_{\pm}) \tag{A.23}$$

for some vectors of complex constants $\boldsymbol{C}_{\pm}$.

When semiclassical decoding holds, one recovers the exact information by just considering Borel-resummed trans-series. Conversely, we can think about resummed trans-series as building blocks of non-perturbative answers.

The simplest situation corresponds to the case in which $C = 0$, there are no singularities along the positive real axis, and the Borel resummation of the perturbative series reproduces the exact result. This is famously the case for the perturbative series of the quartic oscillator, as we mentioned in section 1.

B Faddeev's quantum dilogarithm

Faddeev's quantum dilogarithm $\Phi_{\mathsf{b}}(x)$ is defined by [44]

$$\Phi_{\mathsf{b}}(x) = \frac{(\mathrm{e}^{2\pi \mathsf{b}(x + c_{\mathsf{b}})}; q)_\infty}{(\mathrm{e}^{2\pi \mathsf{b}^{-1}(x - c_{\mathsf{b}})}; \tilde{q})_\infty}, \tag{B.1}$$

where

$$q = \mathrm{e}^{2\pi \mathrm{i} \mathsf{b}^2}, \qquad \tilde{q} = \mathrm{e}^{-2\pi \mathrm{i} \mathsf{b}^{-2}}, \qquad \mathrm{Im}(\mathsf{b}^2) > 0 \tag{B.2}$$

and

$$c_{\mathsf{b}} = \frac{\mathrm{i}}{2} \left(\mathsf{b} + \mathsf{b}^{-1} \right). \tag{B.3}$$

Explicitly, this gives

$$\Phi_{\mathsf{b}}(x) = \prod_{n=0}^{\infty} \frac{1 - \mathrm{e}^{2\pi \mathsf{b}(x + c_{\mathsf{b}})} q^n}{1 - \mathrm{e}^{2\pi \mathsf{b}^{-1}(x - c_{\mathsf{b}})} \tilde{q}^n}. \tag{B.4}$$

From this infinite product representation one deduces that $\Phi(x)$ is a meromorphic function of x with

$$\text{poles: } c_{\mathsf{b}} + \mathrm{i}\mathbb{N}\mathsf{b} + \mathrm{i}\mathbb{N}\mathsf{b}^{-1}, \qquad \text{zeros: } -c_{\mathsf{b}} - \mathrm{i}\mathbb{N}\mathsf{b} - \mathrm{i}\mathbb{N}\mathsf{b}^{-1}. \tag{B.5}$$

An integral representation in the strip $|\mathrm{Im} z| < |\mathrm{Im}\, c_{\mathsf{b}}|$ is given by

$$\Phi_{\mathsf{b}}(x) = \exp \left(\int_{\mathbb{R} + \mathrm{i}\epsilon} \frac{\mathrm{e}^{-2\mathrm{i}xz}}{4 \sinh(z\mathsf{b}) \sinh(z\mathsf{b}^{-1})} \frac{\mathrm{d}z}{z} \right). \tag{B.6}$$

Remarkably, this function admits an extension to all values of b with $\mathsf{b}^2 \notin \mathbb{R}_{\leq 0}$. A useful property is

$$\Phi_{\mathsf{b}}(x)\,\Phi_{\mathsf{b}}(-x) = \mathrm{e}^{\pi \mathrm{i} x^2}\,\Phi_{\mathsf{b}}(0)^2, \qquad \Phi_{\mathsf{b}}(0) = \left(\frac{q}{\tilde{q}}\right)^{\frac{1}{48}} = \mathrm{e}^{\pi \mathrm{i}\left(\mathsf{b}^2+\mathsf{b}^{-2}\right)/24}. \tag{B.7}$$

In addition, when b is either real or on the unit circle, we have the unitarity relation

$$\overline{\Phi_{\mathsf{b}}(x)} = \frac{1}{\Phi_{\mathsf{b}}\left(\overline{x}\right)}. \tag{B.8}$$

From the product representation (B.4) it follows immediately that Faddeev's quantum dilogarithm is a quasi-periodic function. Explicitly, it satisfies the equations

$$\frac{\Phi_{\mathsf{b}}(x + c_{\mathsf{b}} + \mathrm{i}\mathsf{b})}{\Phi(x + c_{\mathsf{b}})} = \frac{1}{1 - q\mathrm{e}^{2\pi \mathsf{b} x}} \tag{B.9a}$$

$$\frac{\Phi_{\mathsf{b}}(x + c_{\mathsf{b}} + \mathrm{i}\mathsf{b}^{-1})}{\Phi_{\mathsf{b}}(x + c_{\mathsf{b}})} = \frac{1}{1 - \tilde{q}^{-1}\mathrm{e}^{2\pi \mathsf{b}^{-1} x}}. \tag{B.9b}$$

When b is small, we can use the folllowing asymptotic expansion,

$$\log \Phi_{\mathsf{b}}\left(\frac{x}{2\pi \mathsf{b}}\right) \sim \sum_{k=0}^{\infty} \left(2\pi \mathrm{i} \mathsf{b}^2\right)^{2k-1} \frac{B_{2k}(1/2)}{(2k)!} \mathrm{Li}_{2-2k}(-\mathrm{e}^x), \tag{B.10}$$

where $B_{2k}(z)$ is the Bernoulli polynomial.

When

$$\mathsf{b}^2 = \frac{M}{N} \tag{B.11}$$

is a rational number, Faddeev's quantum dilogarithm can be written in terms of the conventional dilogarithm [120]. In particular, when $M = N = 1$, one finds

$$\Phi_1(x) = \exp\left[\frac{\mathrm{i}}{2\pi}\left(\mathrm{Li}_2\left(\mathrm{e}^{2\pi x}\right) + 2\pi x \log\left(1 - \mathrm{e}^{2\pi x}\right)\right)\right]. \tag{B.12}$$

References

[1] E. Caliceti, M. Meyer-Hermann, P. Ribeca, A. Surzhykov and U. D. Jentschura, *From useful algorithms for slowly convergent series to physical predictions based on divergent perturbative expansions*, *Phys. Rept.* **446** (2007) 1–96, [0707.1596].

[2] G. 't Hooft, *Can we make sense out of quantum chromodynamics?*, *Subnucl. Ser.* **15** (1979) 943.

[3] S. H. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, *Nucl.Phys.* **B497** (1997) 173–195, [hep-th/9609239].

[4] R. Dijkgraaf, H. L. Verlinde and E. P. Verlinde, *Topological strings in $d < 1$*, *Nucl. Phys. B* **352** (1991) 59–86.

[5] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño and C. Vafa, *Topological strings and integrable hierarchies*, *Commun.Math.Phys.* **261** (2006) 451–516, [hep-th/0312085].

[6] N. A. Nekrasov and S. L. Shatashvili, *Quantization of integrable systems and four dimensional gauge theories*, in *16th International Congress on Mathematical Physics, Prague, August 2009, 265-289, World Scientific 2010*, 2009. 0908.4052.

[7] A. B. Goncharov and R. Kenyon, *Dimers and cluster integrable systems*, 1107.5588.

[8] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings, and supersymmetric gauge theories*, *Nucl. Phys.* **B644** (2002) 3–20, [hep-th/0206255].

[9] V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti, *Remodeling the B-model*, *Commun.Math.Phys.* **287** (2009) 117–178, [0709.1453].

[10] R. Gopakumar and C. Vafa, *On the gauge theory / geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415–1443, [hep-th/9811131].

[11] A. Grassi, Y. Hatsuda and M. Mariño, *Topological strings from quantum mechanics*, *Annales Henri Poincaré* **17** (2016) 3177–3235, [1410.3382].

[12] M. Mariño and S. Zakany, *Matrix models from operators and topological strings*, *Annales Henri Poincaré* **17** (2016) 1075–1108, [1502.02958].

[13] A. Klemm, *The B-model approach to topological string theory on Calabi-Yau n-folds*, in *B-model Gromov-Witten theory* (E. Clader and Y. Ruan, eds.), Springer–Verlag, 2018.

[14] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa et al., *Mirror symmetry*. American Mathematical Society, Providence, USA, 2003.

[15] M. Mariño, *Chern-Simons theory and topological strings*, *Rev.Mod.Phys.* **77** (2005) 675–720, [hep-th/0406005].

[16] M. Vonk, *A Mini-course on topological strings*, hep-th/0504147.

[17] M. Mariño, *An introduction to topological string theory*. https://www.marcosmarino.net/uploads/1/3/3/5/133535336/introts.pdf, 2023.

[18] E. Witten, *Topological Sigma Models*, *Commun. Math. Phys.* **118** (1988) 411.

[19] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637–678, [hep-th/9207094].

[20] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311–428, [hep-th/9309140].

[21] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[22] J. M. F. Labastida and P. M. Llatas, *Topological matter in two-dimensions*, *Nucl. Phys. B* **379** (1992) 220–258, [hep-th/9112051].

[23] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl.Phys.* **B405** (1993) 279–304, [hep-th/9302103].

[24] M. Mariño and G. W. Moore, *Counting higher genus curves in a Calabi-Yau manifold*, *Nucl. Phys. B* **543** (1999) 592–614, [hep-th/9808131].

[25] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, *Inventiones mathematicae* **139** (2000) 173–199, [math/9810173].

[26] R. Gopakumar and C. Vafa, *M-theory and topological strings. 2.*, hep-th/9812127.

[27] S. H. Katz, A. Klemm and C. Vafa, *M theory, topological strings and spinning black holes*, *Adv. Theor. Math. Phys.* **3** (1999) 1445–1537, [hep-th/9910181].

[28] R. Pandharipande and R. P. Thomas, *Stable pairs and BPS invariants*, *Electron. Res. Announ. AMS* **23** (2010) 267–297, [0711.3899].

[29] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *The topological vertex*, *Commun. Math. Phys.* **254** (2005) 425–478, [hep-th/0305132].

[30] N. A. Nekrasov, *Seiberg–Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831–864, [hep-th/0206161].

[31] G. Felder and M. Müller-Lennert, *Analyticity of Nekrasov Partition Functions*, *Commun. Math. Phys.* **364** (2018) 683–718, [1709.05232].

[32] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys. B* **359** (1991) 21–74.

[33] T. Chiang, A. Klemm, S.-T. Yau and E. Zaslow, *Local mirror symmetry: Calculations and interpretations*, *Adv.Theor.Math.Phys.* **3** (1999) 495–565, [hep-th/9903053].

[34] K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222.

[35] M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, JHEP **0803** (2008) 060, [hep-th/0612127].

[36] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun.Num.Theor.Phys. **1** (2007) 347–452, [math-ph/0702045].

[37] M. Aganagic, V. Bouchard and A. Klemm, *Topological Strings and (Almost) Modular Forms*, Commun.Math.Phys. **277** (2008) 771–819, [hep-th/0607100].

[38] D. Ghoshal and C. Vafa, *$c = 1$ string as the topological theory of the conifold*, Nucl. Phys. B **453** (1995) 121–128, [hep-th/9506122].

[39] R. Couso-Santamaría, J. D. Edelstein, R. Schiappa and M. Vonk, *Resurgent transseries and the holomorphic anomaly: Nonperturbative closed strings in local $\mathbb{CP}^2$*, Commun. Math. Phys. **338** (2015) 285–346, [1407.4821].

[40] P. S. Aspinwall, *The Moduli space of N=2 superconformal field theories*, in *ICTP Summer School in High-energy Physics and Cosmology*, pp. 352–401, 11, 1994. hep-th/9412115.

[41] M. Mariño, R. Schiappa and M. Weiss, *Nonperturbative effects and the large-order behavior of matrix models and topological strings*, Commun. Num. Theor. Phys. **2** (2008) 349–419, [0711.1954].

[42] M. Mariño, *Nonperturbative effects and nonperturbative definitions in matrix models and topological strings*, JHEP **0812** (2008) 114, [0805.3033].

[43] S. Pasquetti and R. Schiappa, *Borel and Stokes nonperturbative phenomena in topological string theory and $c = 1$ matrix models*, Annales Henri Poincaré **11** (2010) 351–431, [0907.4082].

[44] L. Faddeev, *Discrete Heisenberg-Weyl group and modular group*, Lett.Math.Phys. **34** (1995) 249–254, [hep-th/9504111].

[45] M.-x. Huang and A. Klemm, *Direct integration for general Ω backgrounds*, Adv. Theor. Math. Phys. **16** (2012) 805–849, [1009.1126].

[46] S. Alexandrov, M. Mariño and B. Pioline, *Resurgence of refined topological strings and dual partition functions*, 2311.17638.

[47] J. Gu, A.-K. Kashani-Poor, A. Klemm and M. Mariño, *Non-perturbative topological string theory on compact Calabi-Yau 3-folds*, SciPost Phys. **16** (2024) 079, [2305.19916].

[48] N. Drukker, M. Mariño and P. Putrov, *Nonperturbative aspects of ABJM theory*, JHEP **11** (2011) 141, [1103.4844].

[49] V. A. Kazakov and I. K. Kostov, *Instantons in noncritical strings from the two matrix model*, in *From Fields to Strings: Circumnavigating Theoretical Physics: A Conference in Tribute to Ian Kogan*, pp. 1864–1894, 3, 2004. hep-th/0403152. DOI.

[50] R. Couso-Santamaría, J. D. Edelstein, R. Schiappa and M. Vonk, *Resurgent transseries and the holomorphic anomaly*, Annales Henri Poincaré **17** (2016) 331–399, [1308.1695].

[51] J. Gu and M. Mariño, *Exact multi-instantons in topological string theory*, SciPost Phys. **15** (2023) 179, [2211.01403].

[52] K. Iwaki and M. Mariño, *Resurgent Structure of the Topological String and the First Painlevé Equation*, SIGMA **20** (2024) 028, [2307.02080].

[53] M. Mariño and M. Schwick, *Large N instantons, BPS states, and the replica limit*, 2403.14462.

[54] H. Dillinger, E. Delabaere and F. Pham, *Résurgence de Voros et périodes des courbes hyperelliptiques*, Annales de l'Institut Fourier **43** (1993) 163–199.

[55] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory*, Nucl. Phys. **B426** (1994) 19–52, [hep-th/9407087].

[56] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. P. Warner, *Selfdual strings and N=2 supersymmetric field theory*, Nucl. Phys. **B477** (1996) 746–766, [hep-th/9604034].

[57] D. Gaiotto, G. W. Moore and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, *Adv. Math.* **234** (2013) 239–403.

[58] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, 0811.2435.

[59] F. Ferrari and A. Bilal, *The Strong coupling spectrum of the Seiberg-Witten theory*, *Nucl. Phys. B* **469** (1996) 387–402, [hep-th/9602082].

[60] M. Mariño, *From resurgence to BPS states*, talk given at the conference *Strings 2019*, Brussels, https://member.ipmu.jp/yuji.tachikawa/stringsmirrors/2019/2_M_Marino.pdf, 2019.

[61] S. Garoufalidis, J. Gu and M. Mariño, *The Resurgent Structure of Quantum Knot Invariants*, *Commun. Math. Phys.* **386** (2021) 469–493, [2007.10190].

[62] S. Garoufalidis, J. Gu and M. Mariño, *Peacock patterns and resurgence in complex Chern-Simons theory*, 2012.00062.

[63] S. Garoufalidis, J. Gu, M. Mariño and C. Wheeler, *Resurgence of Chern-Simons theory at the trivial flat connection*, 2111.04763.

[64] J. Gu and M. Mariño, *Peacock patterns and new integer invariants in topological string theory*, *SciPost Phys.* **12** (2022) 058, [2104.07437].

[65] M. Alim, A. Saha, J. Teschner and I. Tulli, *Mathematical Structures of Non-perturbative Topological String Theory: From GW to DT Invariants*, *Commun. Math. Phys.* **399** (2023) 1039–1101, [2109.06878].

[66] A. Grassi, Q. Hao and A. Neitzke, *Exponential Networks, WKB and Topological String*, *SIGMA* **19** (2023) 064, [2201.11594].

[67] M. Alim, L. Hollands and I. Tulli, *Quantum Curves, Resurgence and Exact WKB*, *SIGMA* **19** (2023) 009, [2203.08249].

[68] S. Alexandrov, J. Manschot, D. Persson and B. Pioline, *Quantum hypermultiplet moduli spaces in N=2 string vacua: a review*, *Proc. Symp. Pure Math.* **90** (2015) 181–212, [1304.0766].

[69] A. Strominger, *Massless black holes and conifolds in string theory*, *Nucl. Phys. B* **451** (1995) 96–108, [hep-th/9504090].

[70] C. Vafa, *A Stringy test of the fate of the conifold*, *Nucl. Phys. B* **447** (1995) 252–260, [hep-th/9505023].

[71] S. Alexandrov, D. Persson and B. Pioline, *Fivebrane instantons, topological wave functions and hypermultiplet moduli spaces*, *JHEP* **03** (2011) 111, [1010.5792].

[72] S. Alexandrov and B. Pioline, *Theta series, wall-crossing and quantum dilogarithm identities*, *Lett. Math. Phys.* **106** (2016) 1037–1066, [1511.02892].

[73] J. Gu and M. Mariño, *On the resurgent structure of quantum periods*, *SciPost Phys.* **15** (2023) 035, [2211.03871].

[74] J. Gu, *Relations between Stokes constants of unrefined and Nekrasov-Shatashvili topological strings*, *JHEP* **05** (2024) 199, [2307.02079].

[75] D. Gaiotto, G. W. Moore and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, *Comm. Math. Phys.* **299** (2010) 163–224.

[76] A. Grassi, J. Gu and M. Mariño, *Non-perturbative approaches to the quantum Seiberg-Witten curve*, *JHEP* **07** (2020) 106, [1908.07065].

[77] S. Hossenfelder, *Lost in math: How beauty leads physics astray.* Basic Books, New York, 2018.

[78] M. Mariño, *Les Houches lectures on matrix models and topological strings*, hep-th/0410165.

[79] E. Witten, *Quantum Field Theory and the Jones Polynomial*, *Commun. Math. Phys.* **121** (1989) 351–399.

[80] A. Kapustin, B. Willett and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **1003** (2010) 089, [0909.4559].

[81] V. Pestun et al., *Localization techniques in quantum field theories*, *J. Phys.* **A50** (2017) 440301, [1608.02952].

[82] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *Matrix model as a mirror of Chern-Simons theory*, *JHEP* **0402** (2004) 010, [hep-th/0211098].

[83] G. Borot and A. Brini, *Chern-Simons theory on spherical Seifert manifolds, topological strings and integrable systems*, *Adv. Theor. Math. Phys.* **22** (2018) 305–394, [1506.06887].

[84] S. Codesido, A. Grassi and M. Mariño, *Spectral theory and mirror curves of higher genus*, *Annales Henri Poincaré* **18** (2017) 559–622, [1507.02096].

[85] M.-x. Huang and X.-f. Wang, *Topological strings and quantum spectral problems*, *JHEP* **1409** (2014) 150, [1406.6178].

[86] R. Kashaev and M. Mariño, *Operators from mirror curves and the quantum dilogarithm*, *Commun. Math. Phys.* **346** (2016) 967, [1501.01014].

[87] R. Kashaev, M. Mariño and S. Zakany, *Matrix models from operators and topological strings, 2*, *Annales Henri Poincaré* **17** (2016) 2741–2781, [1505.02243].

[88] A. Laptev, L. Schimmer and L. A. Takhtajan, *Weyl type asymptotics and bounds for the eigenvalues of functional-difference operators for mirror curves*, *Geom. Funct. Anal.* **26** (2016) 288–305, [1510.00045].

[89] B. Simon, *Notes on infinite determinants of Hilbert space operators*, *Advances in Mathematics* **24** (1977) 244–273.

[90] A. Voros, *Spectral Functions, Special Functions and Selberg Zeta Function*, *Commun. Math. Phys.* **110** (1987) 439.

[91] N. Drukker, M. Mariño and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511–563, [1007.3837].

[92] M. Mariño and P. Putrov, *ABJM theory as a Fermi gas*, *J.Stat.Mech.* **1203** (2012) P03001, [1110.4066].

[93] Y. Hatsuda, M. Mariño, S. Moriyama and K. Okuyama, *Non-perturbative effects and the refined topological string*, *JHEP* **1409** (2014) 168, [1306.1734].

[94] J. Kallen and M. Mariño, *Instanton effects and quantum spectral curves*, *Annales Henri Poincaré* **17** (2016) 1037–1074, [1308.6485].

[95] X. Wang, G. Zhang and M.-x. Huang, *New exact quantization condition for toric Calabi–Yau geometries*, *Phys. Rev. Lett.* **115** (2015) 121601, [1505.05360].

[96] S. Zakany, *Matrix models for topological strings: exact results in the planar limit*, 1810.08608.

[97] M. Mariño, *Spectral Theory and Mirror Symmetry*, *Proc. Symp. Pure Math.* **98** (2018) 259, [1506.07757].

[98] F. Bornemann, *On the numerical evaluation of Fredholm determinants*, *Math. Comp.* **79** (2010) 871–915, [0804.2543].

[99] K. Okuyama and S. Zakany, *TBA-like integral equations from quantized mirror curves*, *JHEP* **03** (2016) 101, [1512.06904].

[100] A. Kapustin, B. Willett and I. Yaakov, *Nonperturbative Tests of Three-Dimensional Dualities*, *JHEP* **1010** (2010) 013, [1003.5694].

[101] Y. Emery, M. Mariño and M. Ronzani, *Resonances and PT symmetry in quantum curves*, *JHEP* **04** (2020) 150, [1902.08606].

[102] I. K. Kostov, *Exact solution of the six vertex model on a random lattice*, *Nucl. Phys.* **B575** (2000) 513–534, [hep-th/9911023].

[103] M. Mariño and S. Zakany, *Exact eigenfunctions and the open topological string*, *J. Phys.* **A50** (2017) 325401, [1606.05297].

[104] E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, *Planar Diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

[105] M. Mariño, *Lectures on non-perturbative effects in large N gauge theories, matrix models and strings*, *Fortsch. Phys.* **62** (2014) 455–540, [1206.6272].

[106] Y. Hatsuda and M. Mariño, *Exact quantization conditions for the relativistic Toda lattice*, *JHEP* **05** (2016) 133, [1511.02860].

[107] S. Franco, Y. Hatsuda and M. Mariño, *Exact quantization conditions for cluster integrable systems*, *J. Stat. Mech.* **1606** (2016) 063107, [1512.03061].

[108] P. L. del Ángel R., C. Doran, M. Kerr, J. Lewis, J. Iyer, S. Müller-Stach et al., *Specialization of cycles and the K-theory elevator*, *Commun. Number Theory Phys.* **13** (2019) 299–349.

[109] C. F. Doran, M. Kerr and S. S. Babu, *K_2 and quantum curves*, 2110.08482.

[110] S. S. Babu, *Integral Regulators on Mirror Curves with Mass Parameter*, 2407.18356.

[111] G. Bonelli, A. Grassi and A. Tanzini, *Seiberg–Witten theory as a Fermi gas*, *Lett. Math. Phys.* **107** (2017) 1–30, [1603.01174].

[112] G. Bonelli, A. Grassi and A. Tanzini, *New results in $\mathcal{N} = 2$ theories from non-perturbative string*, *Annales Henri Poincare* **19** (2018) 743–774, [1704.01517].

[113] P. Gavrylenko, A. Grassi and Q. Hao, *Connecting topological strings and spectral theory via non-autonomous Toda equations*, 2304.11027.

[114] R. Couso-Santamaría, M. Mariño and R. Schiappa, *Resurgence matches quantization*, *J. Phys.* **A50** (2017) 145402, [1610.06782].

[115] T. M. Seara and D. Sauzin, *Resumació de Borel i teoria de la ressurgencia*, *Butl. Soc. Catalana Mat.* **18** (2003) 131–153.

[116] I. Aniceto, G. Basar and R. Schiappa, *A Primer on Resurgent Transseries and Their Asymptotics*, *Phys. Rept.* **809** (2019) 1–135, [1802.10441].

[117] C. Mitschi and D. Sauzin, *Divergent series, summability and resurgence. I*, vol. 2153 of *Lecture Notes in Mathematics*. Springer, 2016, 10.1007/978-3-319-28736-2.

[118] M. Mariño, *Instantons and large N. An introduction to non-perturbative methods in quantum field theory*. Cambridge University Press, 2015.

[119] M. Mariño, *An introduction to resurgence in quantum theory*. https://www.marcosmarino.net/uploads/1/3/3/5/133535336/resurgence-course.pdf, 2023.

[120] S. Garoufalidis and R. Kashaev, *Evaluation of state integrals at rational points*, *Commun. Num. Theor. Phys.* **09** (2015) 549–582, [1411.6062].