GAUGE THEORY INVITATION TO QUANTUM GEOMETRY

or

THE COUNT OF INSTANTONS

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August 12, 12, 14, 16

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Natural studies can be roughly classified into active and passive

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Natural studies can be roughly classified into active and passive

- In active approach we try to interfere with Nature and see what happens

- In passive approach we sit back, observe, and reflect

Sometimes we reflect on the effects of our own prior interference

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The story of gauge theory as the theory of fundamental forces

is a combination of both

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The complicated question of how we perceive NOW

is pushed into the pile of questions on computing the boundary value

of analytic function of complex metric on spacetime

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QUANTUM FIELD THEORY

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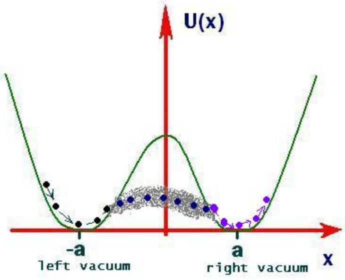
Reconstructs reality with the flow of time

from statistics of events on a four dimensional Riemannian manifold



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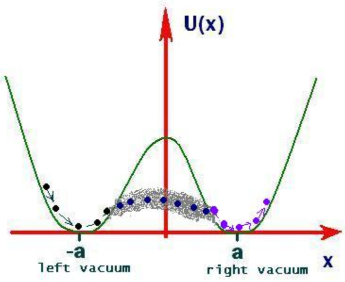
We shall illustrate this point

by studying non-peturbative effects in Yang-Mills theory

field theory analogues of tunneling

Following *Callan, Gross, Dashen, 't Hooft, Polyakov, Coleman*



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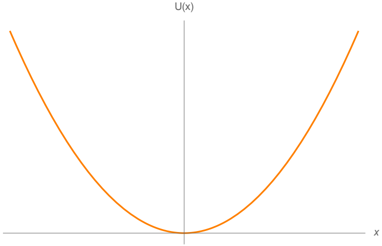
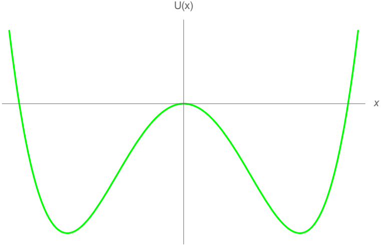
Maxwell theory of electromagnetism

is a field theory version of harmonic oscillator



Anharmonic analogue of Maxwell theory



YANG-MILLS THEORY

$$S = \int_{\mathbb{M}^4} \frac{1}{4g^2} \mathrm{Tr} F_A \wedge \star F_A + \frac{\vartheta}{2\pi} \mathrm{Tr} F_A \wedge F_A$$

$$F_A = dA + \frac{1}{2}[A, A]$$

A is a connection 1-form, locally $A \in \Omega^1(\mathbb{M}^4) \otimes \mathfrak{g}$

We shall discuss gauge groups $G = U(N)$

$\mathfrak{g} = Lie(G)$ (anti-)hermitian $N \times N$ matrices

What is effective field theory

for YANG-MILLS THEORY?

Too difficult in $4d$: Dimensional reduction from $4d$ to $3d$

$$\mathbf{A} \longrightarrow (\mathbf{A}, \Phi)$$

$$S_3 = \frac{1}{4g^2} \int_{M^3} \mathrm{Tr} F_A \wedge \star F_{\mathbf{A}} + \mathrm{Tr} D_{\mathbf{A}} \Phi \wedge \star D_{\mathbf{A}} \Phi$$

Gauge group effectively breaks down to the maximal torus $T \subset G$

One can show, that effective dynamics is that of T-gauge theory

$$\mathbf{A} \longrightarrow A$$

$$S_{eff} = \frac{1}{4g^2}\int_{M^3} dA \wedge \star dA + \int_{M^3} \Lambda^2 \cos(a_D)$$

where the potential $\cos(a_D)$ generates the effects of magnetic monopoles

Polyakov

$$da_D = \star_3 dA$$

In the microscopic theory monopoles are non-singular solutions, representing (complex) saddles, 't Hooft-Polyakov monopoles

Bogomolny equations $\qquad D_{\mathbf{A}}\Phi = \star F_{\mathbf{A}}$

Far from the core look like solutions to Maxwell equations on a manifold of nontrivial topology, with non-contractible S^2 the details are encoded in Λ^2

MAGNETIC FRAME

$$S_{eff} = \frac{g^2}{2} \int_{M^3} da_D \wedge \star da_D + \int_{M^3} \Lambda^2 \cos(a_D)$$

The dual photon a_D is massive

electric charges are confined

$$\langle e^{\oint_C a} \rangle = \langle e^{\int_\Sigma \star da_D} \rangle \sim e^{-\sigma_\Lambda Area(\Sigma)} \quad \partial\Sigma = C$$

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Example of exact computation in $4d$?

Interacting non-abelian gauge theory

Learn about effective description at low energy

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Supersymmetric Yang-Mills theory

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Supersymmetric Yang-Mills theory

Quantum field theory on non-commutative space-time

Coordinate functions obey $[x^\mu, x^\nu] = 0$, $\{\vartheta^\alpha, \vartheta^\beta\} = 0$

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Supersymmetric Yang-Mills theory

$$\mathcal{A}(x,\vartheta) = \Phi(x) + \vartheta^{\alpha}\lambda_{\alpha}(x) + \vartheta^{\alpha}\vartheta^{\dot\beta}(\sigma^{\mu})_{\alpha\dot\beta}A_{\mu}(x) + \dots$$

Fields of different spin are packaged together

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Supersymmetric Yang-Mills theory

$$S = \int_{\mathbb{M}^4} \frac{1}{4g^2} \mathrm{Tr} F_A \wedge \star F_A + \frac{\vartheta}{2\pi} \mathrm{Tr} F_A \wedge F_A +$$

$$\int_{\mathbb{M}^4} \frac{1}{4g^2} \left(\mathrm{Tr} D_A \phi \wedge \star D_A \bar{\phi} + \mathrm{Tr} [\phi, \bar{\phi}]^2 \right) +$$

$$+ \int_{\mathbb{M}^4} \mathrm{Tr} \bar{\psi} \not{D}_A \psi + \mathrm{Tr} \left(\psi [\bar{\phi}, \psi] + \bar{\psi} [\phi, \bar{\psi}] \right)$$

In the minimal $\mathcal{N} = 2$ theory:

the bosons are the gauge field A and the complex adjoint scalar $\phi, \bar{\phi}$
the fermions are adjoint valued ψ and $\bar{\psi}$ Weyl spinors of opposite chirality

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Field theory analogues of saddle complex trajectories

INSTANTONS

Finite action solutions to

$$F_A = - \star F_A$$

real in Euclidean spacetime $\mathbb{R}^4$

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INSTANTONS

The simplest solution can be found by the ansatz

$$A = f(r) g^{-1} dg \,, \ \ g : S^3 \to G$$

In radial coordinates on $\mathbb{R}^4 \backslash 0 = \mathbb{R}_+ \times S^3$

The radial evolution is equivalent to the anharmonic oscillator

$$\frac{1}{2}\left(r \frac{df}{dr} \right)^2 + \frac{1}{4} f^2 (1-f)^2$$

Belavin – Polyakov – Schwarz – Tyupkin solution

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INSTANTONS

Typical instanton solution looks like a non-linear superposition

of k localized objects - events - instances

$$-\frac{1}{8\pi^2}\int_{\mathbb{M}^4} \mathrm{Tr}\, F_{\mathbf{A}} \wedge F_{\mathbf{A}} = k$$

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INSTANTONS of charge k

$$-\frac{1}{8\pi^2}\int_{\mathbb{M}^4} \mathrm{Tr} F_A \wedge F_A = k \in \mathbb{Z}$$

Solutions have parameters (moduli)

$$\mathrm{M}_k(N) = \left\{ A \,|\, F_A = - \star F_A, \int \mathrm{Tr} F_A^2 = -8\pi^2 k \right\} / \mathcal{G}_\infty$$

$\mathcal{G}_\infty$: group of gauge transformations $A \mapsto g^{-1} A g + g^{-1} dg$

$$g(x) \to 1\,, \ x \to \infty$$

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INSTANTONS moduli space

Remarkably, $\mathrm{M}_k(N)$ = complexified phase space

Atiyah — Hitchin — Drinfeld — Manin

of some auxiliary classical mechanical system

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INSTANTONS DOMINATE

With some work, $\mathcal{N} = 2$ susy gauge theory path integral

exactly reduces to the sum of integrals

$$\mathcal{Z}_k = \int_{\mathsf{M}_k(N)} \Theta_k$$

of d-closed differential forms Θ_k,
obtained by careful expansion of S around instanton solutions

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Some steps : • Ω-deformation + Higgsing

Rotational *Spin*(4)-symmetry + constant gauge transformations

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Some steps : Localization

Fixed point formulas

$$\mathcal{Z}_k(\mathbf{a}, \varepsilon_1, \varepsilon_2) = \sum_{\lambda} \mu_{\lambda}(\mathbf{a}, \varepsilon_1, \varepsilon_2)$$

where the sum is over collections

$$\lambda = \left(\lambda^{(1)}, \ldots, \lambda^{(N)} \right) \ , \ \sum_{\alpha=1}^{N} |\lambda^{(\alpha)}| = k$$

of Young diagrams, representing fixed points, i.e. zeroes of $\mathcal{V}$

Some steps : Localization on partly compactified $\mathrm{M}_k(N)$

Uses noncommutative deformation

$$[x^\mu, x^\nu] = \mathrm{i}\Theta^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \star A_\nu - A_\nu \star A_\mu$$

NN+A. Schwarz

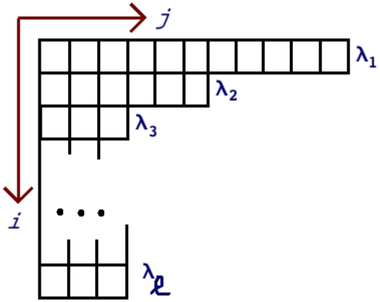
Mathematically, replace vector bundles by sheaves

Gieseker, Nakajima

Fixed point formulas: the sum over collections

$$\lambda = \left(\lambda^{(1)}, \ldots, \lambda^{(N)}\right) \, , \; \sum_{\alpha=1}^{N} |\lambda^{(\alpha)}| = k$$

of Young diagrams, representing fixed points, i.e. zeroes of $\mathcal{V}$, on $\mathrm{M}_k(N)$



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2-Gamma function

$$\Gamma_2(a;\varepsilon_1,\varepsilon_2) \sim \prod_{i,j=1}^{\infty} (a+\varepsilon_1(i-1)+\varepsilon_2(j-1)) =$$

$$= \exp \left.\frac{d}{ds}\right|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty t^s \frac{dt}{t} \frac{e^{-ta}}{(1-e^{-t\varepsilon_1})(1-e^{-t\varepsilon_2})}$$

Functional equation

$$\frac{\Gamma_2(a;\varepsilon_1,\varepsilon_2)\Gamma_2(a+\varepsilon_1+\varepsilon_2;\varepsilon_1,\varepsilon_2)}{\Gamma_2(a+\varepsilon_1;\varepsilon_1,\varepsilon_2)\Gamma_2(a+\varepsilon_2;\varepsilon_1,\varepsilon_2)} = \frac{\Lambda}{a}$$

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Fixed point contribution

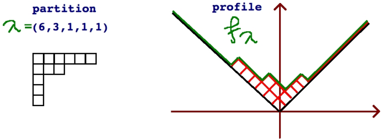
$$\mu_\lambda(\mathbf{a}, \varepsilon_1, \varepsilon_2) = \exp \frac{1}{4} \int \int dx_1 dx_2 f_\lambda''(x_1) f_\lambda''(x_2) \gamma_{\varepsilon_1,\varepsilon_2}(x_1 - x_2)$$

NN+A.Okounkov

is one-loop computation in quantum field theory

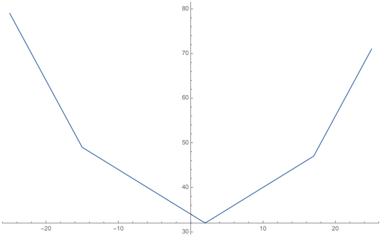
$$\gamma_{\varepsilon_1,\varepsilon_2}(x) = \log \Gamma_2(x; \varepsilon_1, \varepsilon_2)$$

Mathematically: product of weights of $T_{fixed\ point} M_k(N)$



Profile of partition(s)

Examples for $N = 3$



Profile of partition(s)

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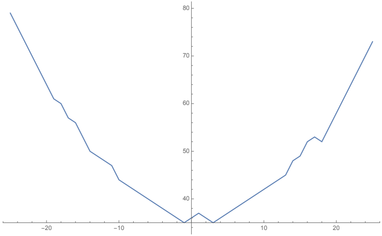

Profile of partition(s)

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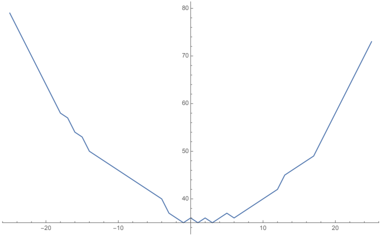

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Profile of partition(s) and instanton measure

$$\mu_\lambda = \prod_{\alpha,\beta=1}^{N} K_{\lambda^{(\alpha)},\lambda^{(\beta)}}(a_\alpha - a_\beta; \varepsilon_1, \varepsilon_2)$$

$$K_{\lambda,\mu}(a) = \Gamma_2(a; \varepsilon_1, \varepsilon_2) \times \prod_{\square=(i,j)\in\lambda} \frac{1}{a + \varepsilon_1(\mu_i - j) + \varepsilon_2(i + 1 - \lambda_j^t)} \times$$

$$\times \prod_{\blacksquare=(i',j')\in\mu} \frac{1}{a + \varepsilon_1(i' + 1 - \mu_{j'}^t) + \varepsilon_2(\lambda_{i'} - j')}$$

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The path to effective theory

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The path to effective theory

Compute the susy partition function

$$\mathcal{Z}(\mathbf{a}, \varepsilon_1, \varepsilon_2; \Lambda) = \sum_{k=0}^{\infty} \Lambda^{2Nk} \sum_{\lambda,\, |\lambda|=k} \mu_\lambda(\mathbf{a}, \varepsilon_1, \varepsilon_2)$$

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The path to effective theory

Small $\varepsilon_1, \varepsilon_2$ asymptotics of susy partition function

$$\mathcal{Z}(\mathbf{a}, \varepsilon_1, \varepsilon_2; \Lambda) = \exp \frac{1}{\varepsilon_1 \varepsilon_2} \mathcal{F}(\mathbf{a}; \Lambda) +$$

sub-leading terms in $\varepsilon_1, \varepsilon_2$

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The low-energy effective theory

$$\mathcal{S}_{eff} = \int_{\mathbb{M}^4} \tau_{\alpha\beta}(\mathbf{a}) F^{\alpha,-} \wedge F^{\beta,-} - \bar{\tau}_{\alpha\beta}(\bar{\mathbf{a}}) F^{\alpha,+} \wedge F^{\beta,+} +$$

$$+ \int_{\mathbb{M}^4} \mathrm{Im}\tau_{\alpha\beta}(\mathbf{a}, \bar{\mathbf{a}}) d a^{\alpha} \wedge \star d \bar{a}^{\beta} +$$

fermions

$$\tau_{\alpha\beta} = \frac{\partial^2 \mathcal{F}(\mathbf{a}; \Lambda)}{\partial a^{\alpha} \partial a^{\beta}}$$

describes $N-1$ photons A^{α},
interacting with $N-1$ complex massless scalars a^{α}

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Seiberg-Witten geometry

$$\mathcal{S}_{eff} = \int_{\mathbb{M}^4} \tau_{\alpha\beta}(\mathbf{a}) F^{\alpha,-} \wedge F^{\beta,-} - \bar{\tau}_{\alpha\beta}(\bar{\mathbf{a}}) F^{\alpha,+} \wedge F^{\beta,+} +$$

$$+ \int_{\mathbb{M}^4} \mathrm{Im}\tau_{\alpha\beta}(\mathbf{a}, \bar{\mathbf{a}}) d a^{\alpha} \wedge \star d \bar{a}^{\beta} +$$

fermions

$$\tau_{\alpha\beta} = \frac{\partial^2 \mathcal{F}(\mathbf{a}; \Lambda)}{\partial a^{\alpha} \partial a^{\beta}}$$

with holomorphic $\mathcal{F}$ cannot describe unitary theory for all $\mathbf{a}$ as kinetic term for scalars and effective couplings for photons cannot be everywhere positive definite

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Seiberg-Witten geometry

way out: electric-magnetic duality

$$\mathbf{a} \mapsto \mathbf{a}_D = \frac{\partial \mathcal{F}}{\partial \mathbf{a}}\,, \;\; F^{\alpha,-} \mapsto \tau_{\alpha\beta} F^{\beta,-}\,, \;\; \tau \mapsto -\tau^{-1}$$

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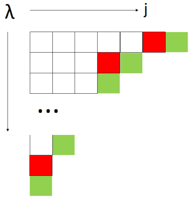
emergent Seiberg-Witten geometry

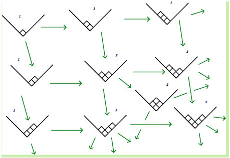
DEFINE the $Y(x)$ observables

$$Y(x)|_{\lambda} = \prod_{\alpha=1}^{N} \frac{\prod_{(i,j)=\blacksquare\in\partial_{+}\lambda^{(\alpha)}} (x - a_{\alpha} - \varepsilon_1(i-1) - \varepsilon_2(j-1))}{\prod_{(i',j')=\blacksquare\in\partial_{-}\lambda^{(\alpha)}} (x - a_{\alpha} - \varepsilon_1 i' - \varepsilon_2 j')}$$



Non-perturbative Dyson-Schwinger equations

$$\left\langle Y(x+\varepsilon_1+\varepsilon_2) + \frac{\Lambda^{2N}}{Y(x)} \right\rangle \text{ has no poles in } x$$



DYSON-SCHWINGER: INVARIANCE OF (PATH) INTEGRAL

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \frac{1}{Z} \int_\Gamma D\Phi \, e^{-\frac{1}{\hbar} S[\Phi]} \, \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n)$$

UNDER “SMALL” DEFORMATIONS
OF THE INTEGRATION CONTOUR

$$\Phi \longrightarrow \Phi + \delta\Phi$$

DYSON-SCHWINGER EQUATIONS

QUANTUM EQUATIONS OF MOTION

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\delta S[\Phi]\rangle =$$
$$\hbar\sum_{i=1}^{n}\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_{i-1}(x_{i-1})\delta\mathcal{O}_i(x_i)\mathcal{O}_{i+1}(x_{i+1})\dots\mathcal{O}_n(x_n)\rangle$$

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DYSON-SCHWINGER EQUATIONS

WITH SOME LUCK

$=$

GOOD CHOICE OF (POSSIBLY NON-LOCAL) OBSERVABLES

$$\mathcal{O}_i(x)$$

AND IN SOME LIMIT (CLASSICAL, PLANAR, ...)

THE DS EQUATIONS FORM A CLOSED SYSTEM

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MATRIX MODEL

$$\int_{N\times N} d\Phi e^{-\frac{1}{\hbar}\mathrm{Tr}V(\Phi)}$$

PLANAR LIMIT: $\lambda = \hbar N$ FIXED

$$\hbar \to 0, \;\; N \to \infty$$

DS eqs $\Longrightarrow$ LOOP EQUATIONS

Define $y(x) = \langle \mathrm{Tr}\dfrac{1}{x-\Phi}\rangle + V'(x)$

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MATRIX MODEL DS EQUATIONS

$$y(x)^2 = V'(x)^2 + g_{p-2}(x)$$

$$g_{p-2}(x) = \text{deg } p-2 \text{ polynomial in } x$$

CLASSICAL GEOMETRY

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QFT PATH INTEGRAL INVOLVES SUMMATION

OVER TOPOLOGICAL SECTORS

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NON-PERTURBATIVE DS EQUATIONS

IDENTITIES DERIVED BY

Large “DEFORMATIONS” OF THE PATH INTEGRAL CONTOUR

$$A \in \mathcal{A}_k \longrightarrow A + \delta A \in \mathcal{A}_{k+1}$$

GRAFTING A POINT-LIKE INSTANTON

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MAIN CLAIM: "qq-characters"

There are combinations of $Y(x)$'s

such that their expectation values have no poles in x

Non-perturbative Dyson-Schwinger equations!

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Back to our example of $U(N)$ theory

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In the $\varepsilon_1, \varepsilon_2 \to 0$ limit DS equations become algebraic

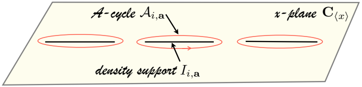
$$Y(x) + \frac{\Lambda^{2N}}{Y(x)} = x^N + u_2 x^{N-2} + \ldots + u_N$$

Parameters $u_2, \ldots, u_N$ are determined from

$$a_\alpha = \frac{1}{2\pi \mathrm{i}} \oint_{A_\alpha} x \frac{dY}{Y}$$

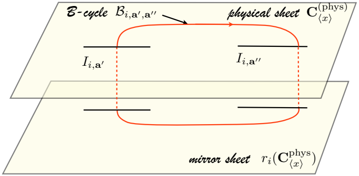
The poles and zeroes of the random function $Y(x)$
Accumulate, in the $\varepsilon_1, \varepsilon_2 \to 0$ limit, to N cuts
Close to a_1, ..., a_N

For theories with gauge groups $G = \times_i U(N_i)$



The poles and zeroes of random functions $Y_i(x)$ accumulate close to $a_{i,1}, \ldots, a_{i,N}$

Algebraic curve: a ramified cover of x-plane



$$a^{\alpha}_{i,D} = \frac{1}{2\pi \mathrm{i}} \oint_{B_{i,\alpha}} x \, \frac{dY_i}{Y_i}$$

Riemann identities

$$\sum_{i,\alpha} d a_{i,\alpha} \wedge d {a_{i,D}}^{\alpha} = 0$$

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Complex symplectic geometry

Therefore, locally at least, there exists the symplectic potential

$$a_{i,D}{}^{\alpha} = \frac{\partial \mathcal{F}}{\partial a_{i,\alpha}}$$

Electric-magnetic duality $=$ choice of A-cycles

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Again, but differently: complex phase spaces

$$Y(x) + \frac{\Lambda^{2N}}{Y(x)} = x^N + u_2 x^{N-2} + \ldots + u_N$$

The family of complex curves $\mathcal{C}_{\mathbf{u}}$, $\mathbf{u} \in \mathbb{C}^{N-1}$

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Integrable complexification of a phase space

$$\sum_{i=1}^{N} dx_i \wedge dz_i \,, \ \sum_i x_i = 0\,, \ (z_i) \sim (z_i + s),$$

$$H_2 = \sum_{i=1}^{N} \frac{1}{2} x_i^2 + \Lambda^2 \sum_{i=1}^{N} e^{z_i - z_{i+1}} \,, \ z_{N+1} \equiv z_1$$

Periodic N-particle Toda chain

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Back to start: complex phase spaces

$$Y(x) + \frac{\Lambda^{2N}}{Y(x)} = x^N + u_2 x^{N-2} + \ldots + u_N$$

Auxiliary linear problem

Has gauge theory origin

$$\psi_{i+1} + x_i \psi_i + \Lambda^2 e^{z_i - z_{i+1}} \psi_{i-1} = x \psi_i$$

$$\psi_{i+N} = Y \psi_i$$

Compatibility of these equations $\Longrightarrow (x, Y) \in \mathcal{C}_{\mathbf{u}}$

Krichever approach

with $u_k = H_k(x, z),\ k = 2, \ldots, N$

$$\{H_k, H_l\} = \sum_i \frac{\partial H_k}{\partial x_i}\frac{\partial H_l}{\partial z_i} - \frac{\partial H_l}{\partial x_i}\frac{\partial H_k}{\partial z_i} = 0$$

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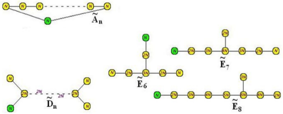
FURTHER DEVELOPMENTS

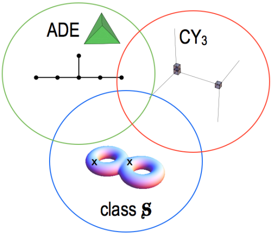
FURTHER DEVELOPMENTS

More general gauge theories with matter



NN+V.Pestun'2012,'2023

String theory realizations



E.Witten, C.Vafa, D.Gaiotto, G.Moore, A.Neitzke, L.Alday, Y.Tachikawa, ...

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FURTHER DEVELOPMENTS

Deeper notions of symmetry, quantization, integrability

Instanton counting on $\mathbb{R}^4$ at $\varepsilon_1 = -\varepsilon_2 = g_s$

$=$ topological string computations on a local Calabi-Yau 3fold

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FURTHER DEVELOPMENTS *Quantizations*

Instanton counting on $\mathbb{R}^4$ at $\varepsilon_1 = \hbar$, $\varepsilon_2 \to 0$

$=$ QUANTUM INTEGRABLE SYSTEMS

$$\widehat{H}_2 = \sum_{i=1}^{N} -\frac{1}{2}\left(\hbar \frac{d}{dz_i}\right)^2 + \Lambda^2 \sum_{i=1}^{N} e^{z_i - z_{i+1}}\,, \;\; z_{N+1} \equiv z_1$$

$For\;\; example\;\; of\;\; pure\;\; super-Yang-Mills$

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Further developments

Quantizations $\varepsilon \neq 0$

Instanton counting on $\mathbb{R}^4$ at $\varepsilon_1, \varepsilon_2 \neq 0$

$=$ Analytically continued conformal field theories in 2d

with S.Jeong, N.Lee, O.Tsymbaliuk

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Further developments

Instanton counting on more general $\mathbb{M}^4$ at $\varepsilon_1, \varepsilon_2 \neq 0$

Predictions/theorems about 2d CFT/Isomonodromic equations

Gamayun−Iorgov−Lysovii "Kyiv" formula

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Further developments

Instanton counting on $S^1 \times \mathbb{M}^4$ at $\varepsilon_1, \varepsilon_2 \neq 0$

$=$ Relativistic integrable systems/q-deformed CFTs/massive IFTs

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Further developments

Instanton counting on $S^1 \times S^1 \times \mathbb{M}^4$ at $\varepsilon_1, \varepsilon_2 \neq 0$

$=$ Double elliptic integrable systems/q-deformed CFTs/massive IFTs

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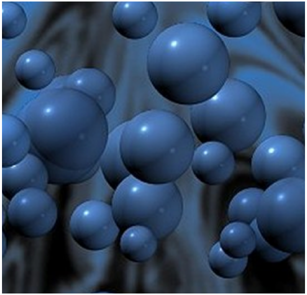
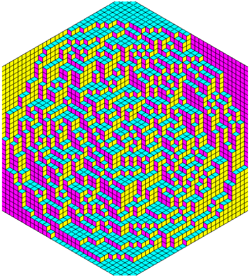
Further developments

Instanton counting on $S^1 \times \mathbb{M}^6$ at $\varepsilon_1, \varepsilon_2, \varepsilon_3 \neq 0$

$=$ Models of crystal melting/quantum spacetime foam

N.Reshetikhin, A.Okounkov, C.Vafa; A.Iqbal, NN, A.Okounkov, C.Vafa





DT-GW correspondence

D.Maulik, NN, A.Okounkov, R.Pandharipande

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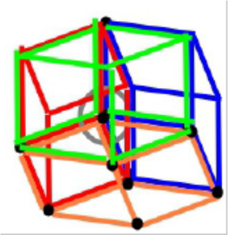
Further developments

Instanton counting on $S^1 \times \mathbb{M}^8$ at $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0$

$=$ Models of $4d$ crystals $/3d$ tesselations

Magnificent Four theory, NN′2017–



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OPEN PROBLEMS and DIRECTIONS of RESEARCH

- Instanton counting for general gauge groups/matter contents

- Hyperkähler geometry of moduli space of vacua for $\mathbb{M}^4 = S^1 \times \mathbb{R}^3$

- Higher category structure (interfaces, stable envelopes, junctions)

  ∞−dim version of Maulik−Okounkov, in progress with M.Dedushenko

- $\mathcal{N} = 1$ theories (periods as opposed to intersection numbers)

- CATEGORIFY $S^1 \longrightarrow \mathbb{R}^1$

OTHER RELATED TOPICS

Syllabus in lieu of abstract

Classical geometry of gauge theory: connections as maps. Instanton connections and holomorphic maps. Classifying spaces: Milnor construction, infinite Grassmanians, Cartan model of de Rham complex. Equivariant cohomology and supersymmetric quantum mechanics. Two dimensional gauge theory, Hurwitz theory and matrix models. Supersymmetric gauge theory in four dimensions, localization to random partitions, comparison to random matrices. Seiberg-Witten geometry, Omega-deformation, quantum integrability, isomonodromic deformations, conformal blocks.