Les Houches lectures on Jackiw-Teitelboim gravity

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Contents

1 Introduction

This is a review of the derivation of some holographic dualities between solvable models of two-dimensional gravity and ensembles of random matrices. I will focus on the all-order perturbative (in the topological expansion) considerations that lead to the dualities.

Due to time constraints I will have to skip some important topics such as the Sachdev-Ye-Kitaev (SYK) model, or the relevance of Jackiw-Teitelboim gravity to higher-dimensional near-extremal black holes.

Several suggested calculations, or intermediate steps to fill, are in red boxes.

Some parts of these lectures overlap with the review written with T. Mertens for Living Reviews in Relativity [1]. Given the mathematical background from the first week, I decided to focus on more technical aspects which were not covered in that review.

2 Lecture 1: Jackiw-Teitelboim Gravity

2.1 Motivation: Holography and the central dogma

In the 1970's it was uncovered that black holes evolve following dynamical laws that take the same form as those of thermodynamics. Perhaps the most famous fact is that black holes have an entropy

$$S = \frac{A}{4G_N}, \tag{2.1}$$

as well as a non-vanishing temperature.

The field of black hole thermodynamics evolved over the decades and culminated in the 1990's with the advent of holography and AdS/CFT. The main lesson, whose implications we are still exploring, is that black holes not only behave in as thermodynamic systems, they actually evolve as unitary quantum systems with large but finite entropy! String theory provided realizations of this relation by assigning specific quantum systems to certain black holes.

"Central dogma: As seen from the outside, a black hole can be described in terms of a quantum system with $S = A/4G_N$ degrees of freedom, which evolves unitarily under time evolution," as stated in [2].

So far the most successful approach to studying quantum gravity is provided by the gravitational path integral, which produces answers consistent with the central dogma in highly non-trivial situations. The gravitational path integral was pioneered by Gibbons and Hawking [3] but it has evolved considerably over the past decades. This consists in formulating the experiment done on the black hole in path integral language, use this to determine boundary conditions far from the horizon, and perform a path integral including fluctuations in the spacetime metric and topology.

The central dogma and the gravitational path integral interpretation are the most precise in the context of AdS/CFT. The boundary conditions in the gravitational path integral are determined at the conformal boundary and according to the holographic dictionary it should reproduced by a CFT calculation without gravity.

We will study in these lectures theories of two-dimensional gravity with black hole solutions, particularly Jackiw-Teitelboim (JT) gravity. It has been useful for three purposes mostly:

- It is a sector describing the low energy dynamics of strongly coupled fermion systems such as the SYK model [4].
- JT gravity with matter captures quantum effects that become large for higher dimensional charged black holes at low temperatures [5–7].
- Black hole information paradox concern situations where gravity seems to be in tension with the central dogma. Examples are the Page curve and the late time behavior of correlators. Thanks to toy models such as JT gravity, it was discovered recently that spacetimes with non-trivial topologies such as spacetime wormholes play a central role in resolving these puzzles. Aspects of this application of JT gravity will be the main focus of this series of lectures.

2.2 Two-dimensional dilaton gravity

The goal is to construct a theory of two-dimensional gravity with a clear semiclassical limit presenting black hole solutions which we can use to explore the consequences of the gravitational path integral.

In two dimensions, the Einstein-Hilbert action is topological and does not suppress fluctuations

$$\chi = \frac{1}{4\pi}\Big(\int_M \sqrt{g}R + 2\oint_{\partial M}\sqrt{h}K\Big) = 2 - 2g - n, \tag{2.2}$$

given by the Euler characteristic. This fact does not imply that the theory is trivial. In the path integral formulation one needs to take care of the measure and its gauge fixing. This leads to an inherently strongly coupled theory[1].

To solve this problem we follow and introduce a scalar field, the dilaton Φ, following [8, 9]. We consider the action, written in Euclidean signature

$$I = \underbrace{-\frac{S_0}{4\pi}\int_M \sqrt{g}R}_{\text{topological}} \underbrace{-\frac{1}{2}\int_M \sqrt{g}(\Phi R + U(\Phi))}_{\text{dynamical}} - \underbrace{\oint_{\partial M}\sqrt{h}\Phi K}_{\text{Gibbons-Hawking-York term}} \tag{2.3}$$

M is a two-dimensional manifold with metric g and with a boundary ∂M with metric h. The action has three terms which play different roles:

[1] To make this more precise, this theory can be put in the context of the non-critical string since a trivial matter CFT can be presented as the $(2,3)$ minimal model. Therefore this theory would be the $(2,3)$ minimal string.

Topological term Depends on the parameter S_0. Responsible to suppress topology change but does not care about perturbative metric fluctuations.

Dynamical term This term is responsible for determining the classical solutions. The scalar field Φ acts as a two-dimensional Planck "mass". The classical limit corresponds to regions where Φ is large, in a way we will see later. This term depends on a single function $U(\Phi)$, the **dilaton potential**.

Boundary term This is the well-known Gibbons-Hawking-York term that makes the variational problem well-defined [3]. It particularly plays an important role in JT gravity, as we will see later.

> **Most general dilaton-gravity**
>
> Show that this is the most general two-dimensional dilaton gravity at the two-derivative level, up to field redefinitions

Jackiw-Teitelboim (JT) gravity corresponds to a theory with a linear dilaton potential

$$U(\Phi) = -\Lambda\Phi + U_0. \tag{2.4}$$

When $\Lambda \neq 0$ we can shift Φ to eliminate U_0, after redefining S_0. After this manipulation the action becomes

$$I_{JT} = -\frac{S_0}{4\pi}\int_M \sqrt{g}R - \frac{1}{2}\int_M \sqrt{g}\Phi(R-\Lambda) - \oint_{\partial M}\sqrt{h}\Phi K \tag{2.5}$$

Some comments:

The equation of motion for the dilaton imposes that classical geometries are spacetimes with constant curvature $R = \Lambda$. The equation of motion for the metric determines the spacetime profile of the dilaton.

We can consider anti deSitter (AdS) gravity with $\Lambda < 0$, or deSitter (dS) gravity with $\Lambda > 0$. When $\Lambda = 0$ we cannot remove U_0 which remains as a parameter. The theory becomes the CGHS model [10]. In these lectures, we will focus mostly on AdS and work in units with $\Lambda = -2$.

One can also include matter fields. In the simplest case they do not couple to the dilaton, e.g. for a massive scalar field we add the following term

$$I_{\text{matter}}[g,\chi] = \frac{1}{2}\int \sqrt{g}\{(\partial\chi)^2 + m^2\chi^2\}. \tag{2.6}$$

When matter is included, we will assume it takes this form. This has the advantage of making the theory solvable as we might see later.

2.3 JT gravity as a BF theory

Let us rewrite the action of JT gravity in the first-order formulation. For simplicity, consider surfaces without boundaries first. This means we want to replace the path integral over the metric $g_{\mu\nu}$ by the objects:

- Frame one-form $e^a = e^a_\mu \, \mathrm{d}x^\mu$ with $a = 1, 2$. They are determined by the metric through the relation $g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$.
- Spin connection $\omega^{ab} = \omega_\mu^{[ab]} \, \mathrm{d}x^\mu$. Its not an independent field since its required to solve the torsion-free constraint $\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$.

The following relations are useful $\omega^{ab} = \epsilon^{ab}\omega$, $\mathrm{d}^2x\,\sqrt{g} = e^1 \wedge e^2$, and $\mathrm{d}^2x\,\sqrt{g}R = 2\,\mathrm{d}\omega$.

The dynamical bulk term in the JT gravity action can be written in terms of the frame and spin connection as

$$\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\Phi(R+2) = \int_M \Phi(\mathrm{d}\omega + e^1 \wedge e^2) \tag{2.7}$$

In a quantum-mechanical treatment we need to incorporate the torsionless constraint that determines ω in terms of the frame forms. This can be remedied by integrating-in Lagrange multipliers X^1 and X^2 as follows

$$\int_M \Big[\Phi(\mathrm{d}\omega + e^1 \wedge e^2) + X_a(\mathrm{d}e^a + \omega^a{}_b \wedge e^b)\Big]. \tag{2.8}$$

Now define the following quantities

$$\begin{aligned} A &= e^1\lambda^1 + e^2\lambda^2 + \omega\lambda^3, \\ B &= 2\mathrm{i}(X^1\lambda^1 + X^2\lambda^2 + \Phi\lambda^3), \end{aligned}$$

where $\{\lambda^1, \lambda^2, \lambda^3\}$ are 2×2 matrices that generate the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, chosen with the normalization condition $\mathrm{Tr}\,\lambda^i\lambda^j = \eta^{ij}/2$ with $\eta = \mathrm{diag}(1, 1, -1)$. The signature of η reflects the fact that $\mathrm{SL}(2, \mathbb{R})$ is a non-compact group.

The factor of 2 in B is convention. The factor of i is such that the contour appropriate to a Lagrange multiplier takes place in the real B axis, or equivalently imaginary Φ.

In terms of these adjoint-valued one-form A and zero-form B, the JT gravity action, including the torsion constraint, can be written in the suggestive form

$$I = -\mathrm{i}\int \mathrm{Tr}\, BF, \qquad F = \mathrm{d}A + A \wedge A. \tag{2.9}$$

This is the action of a BF theory [11] with group $\mathrm{SL}(2, \mathbb{R})$. The path integral of this theory on a surface Σ localizes into the space $\mathcal{T}$ of flat $\mathrm{SL}(2, \mathbb{R})$ connections (with $F = 0$) modulo gauge transformations. The reason is that B acts as a Lagrange multiplier

$$\int \mathrm{d}B\, e^{-I} = \int \mathrm{d}B\; e^{\mathrm{i}\int \mathrm{Tr} BF} = \delta(F). \tag{2.10}$$

What is the measure of integration over moduli space? This will be relevant for lecture 2.

Even though we matched the bulk actions, the connection between JT gravity and BF theory is subtle for the following reasons:

Flat connection $\neq$ geometry The moduli space $\mathcal{T}$ has multiple components distinguished by a topological invariant. Only one of these components can be related to a hyperbolic metric. (This technically includes also a choice of spin structure. To remove this we can focus on $\mathrm{PSL}(2,\mathbb{R})$. More discussion on such global issues later.)

Large Diffeomorphisms Gravity contains large diffeomorphisms as a gauge symmetry, the mapping class group, that are not incorporated into the gauge transformations of the BF description. This restrict the appropriate component of flat connections $\mathcal{T}$ further to the moduli space of hyperbolic surfaces $\mathcal{M}$.

Sum over topologies In gravity we should sum over topologies, as we will do in Lecture 2. This is not naturally included in the gauge theory description but should be done by hand.

Boundaries The boundary conditions natural from the gravity perspective do not have a natural description in the gauge theory language. Therefore in the presence of boundaries, a mix of first- and second-order manipulations seems to be unavoidable.

Poisson Sigma Models

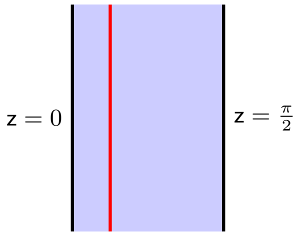
For general dilaton potential show that the gravitational action can be written locally as a Poisson sigma model [12, 13]. The form of the algebra depends explicitly on the dilaton potential.

2.4 Classical Solutions and Boundary Conditions

Before attempting to evaluate the gravitational path integral in JT gravity we need to specify which boundary conditions we want to impose. This requires some physical considerations that we now describe. A lack of understanding of the physically relevant boundary condition was partly the source of confusion regarding AdS_2 in the past.

The equation of motion for the dilaton imposes $R = -2$. Let us assume first that the dilaton is constant. The equation of motion arising from varying the metric, with the assumption of a constant dilaton Φ, implies that $U(\Phi) = 0$ and therefore $\Phi = 0$. This implies that locally all solutions have the metric of two-dimensional AdS_2 space. Globally there can be physically different choices:

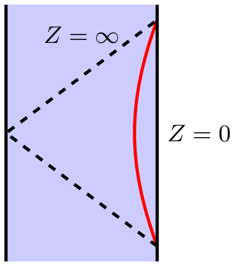
Global Patch In Lorentzian signature, this patch has the topology of a strip with two boundaries



$$ds^2 = 4\frac{-\mathrm{d}t^2 + \mathrm{d}z^2}{\sin^2 2z} \tag{2.11}$$

This patch represents the maximal extension of AdS_2. The red line shows the worldline of an observer sitting at a fixed spatial z. The two boundaries are in causal contact, and can therefore be interpreted as an eternal wormhole connecting the two boundaries.

Poincare Patch In Lorentzian signature this patch has a single boundary and covers a region inside the global patch



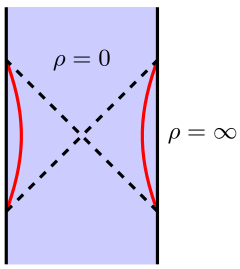
$$ds^2 = \frac{-\mathrm{d}T^2 + \mathrm{d}Z^2}{Z^2} \tag{2.12}$$

The red line corresponds to an observer sitting at a constant Z. This looks closer to a black hole with the dashed line representing the event horizon. There are regions in the bulk that are causally disconnected from the boundary. Nevertheless the horizon is at an infinite proper distance and has zero-temperature, making it the two-dimensional analog of vacuum AdS in higher dimensions. The metric on this patch written above simplifies the action of the $\mathrm{PSL}(2,\mathbb{R})$ isometry group

$$X^{\pm} \to \frac{a\,X^{\pm} + b}{c\,X^{\pm} + d}, \qquad X^{\pm} = T \pm Z, \tag{2.13}$$

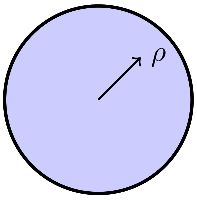
where $ad - bc = 1$ and the four parameters are defined up to an overall sign (which would be detected by a fermion).

Black Hole Patch This patch corresponds truly to a black hole geometry, and it is sometimes called the Rindler patch:



$$ds^2 = -\frac{4\pi^2}{\beta^2}\sinh^2\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2. \tag{2.14}$$

The red lines are worldlines of observers sitting at fixed ρ. The dashed lines are event horizon for observers at both boundaries. Therefore, there are points in the bulk that are causally disconnected, and the horizon is at a finite distance from the boundary (in our coordinates corresponds to $\rho = 0$), and has a finite temperature given by $1/\beta$. In Euclidean signature, the geometry becomes the hyperbolic disk with the horizon at the origin



$$ds^2 = \sinh^2\rho\, d\theta^2 + d\rho^2, \quad \theta = \frac{2\pi t_E}{\beta}.$$

Euclidean

According to holography, quantum gravity in AdS_2 should be dual to a quantum mechanical theory living on the boundary. Since the Rindler patch comes with two boundaries there should be two copies of that theory in some specific state. The gravitational path integral over half the disk prepares the so-called thermo-field double [14]

$$|\mathrm{TFD}\rangle = \sum_n e^{-\beta E_n/2} |E_n\rangle_L \otimes |E_n\rangle_R \tag{2.15}$$

and the path integral over the full disk computes the overlap

$$\begin{aligned}\langle \mathrm{TFD}|\mathrm{TFD}\rangle &= \sum_n e^{-\beta E_n} = \mathrm{Tr}(e^{-\beta H}), \\ &= Z(\beta),\end{aligned} \tag{2.16}$$

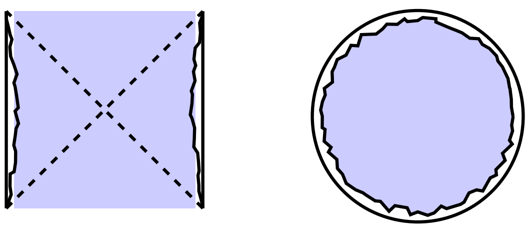
which can also be interpreted as the thermal partition function in the canonical ensemble of the putative quantum system describing the black hole. This is clear in Euclidean signature; the boundary of the hyperbolic disk is a circle and the path integral of a quantum system on a circle is a thermal partition function.

So far we assumed that the dilaton was constant $\Phi = 0$. But this is problematic. Which patch are we supposed to choose? What determines the relation between bulk time and boundary time in the quantum mechanical description?

The solution [15–18] is to use boundary conditions that break the conformal symmetry of AdS_2 by turning on a source for the dilaton. This is something special that happens in two dimensions; in higher-dimensional version of AdS/CFT it is safe to work with theories that are exactly conformal invariant. To implement this, put a cutoff in the geometry and impose Dirichlet boundary conditions

$$ds^2|_{\partial M} = \frac{dt^2}{\varepsilon^2}, \qquad \Phi|_{\partial M} = \frac{\Phi_r}{\varepsilon}, \quad \varepsilon \to 0. \tag{2.17}$$

This leads to the so-called nearly-AdS space, or $NAdS_2$. The geometry, either in Lorentz or Euclidean signature, will now look like the following, where we draw the cut-off curve in black:



(2.18)

Lorentzian Euclidean

At finite temperature the Rindler patch is selected, or at zero-temperatures the Poincare one. The global patch does not support a solution for the dilaton with the appropriate boundary conditions (unless one deforms the theory in a special way [19]). The geometry can still be taken to be locally exactly AdS everywhere. We can work in a gauge where all the information of metric fluctuations is encoded in the shape of the boundary curve consistent with (2.17).

Symmetries The asymptotic symmetries of AdS_2 are the one-dimensional conformal group of time reparametrizations. This is spontaneously broken to the global conformal group $SL(2,\mathbb{R})$ of isometries by the background. Moreover, the conformal symmetry is explicitly broken by the choice of boundary conditions since Φ_r is dimensionful. Another way to see this is that classically the dilaton is proportional to $\Phi \propto \cosh\rho$ and its manifestly non-invariant.

The Schwarzian mode The off-shell action of a given boundary curve consistent with the choice of boundary conditions in (2.17) is the so-called Schwarzian action, which appeared first in the context of the SYK model [20–23]. This can be done in Lorentzian signature but since most of our calculations in the next section are carried out naturally in Euclidean signature we will choose the latter.

We work in the hyperbolic disk with coordinates $(\hat{t},\hat{\rho})$. The reason to relabel them is that we want to save t for the boundary time. Let us denote the location of the boundary by $(\hat{t},\hat{\rho}) = (\hat{t} = f(t), \hat{\rho} = \rho(t))$. Since Euclidean time is compact the first variable $f(t)$ should satisfy

$$f(t) \in \text{Diff}(S^1), \qquad f(t+\beta) = f(t) + \beta. \tag{2.19}$$

What about $\rho(t)$? This is determined by the Dirichlet boundary condition on the metric

$$\mathrm{d}s^2|_{\text{bdy}} = \Big(\rho'(t)^2 + \frac{4\pi^2}{\beta}\sinh^2\rho(t)\, f'(t)^2\Big)\mathrm{d}t^2 \sim \frac{4\pi^2}{\beta}\frac{e^{2\rho}}{4} f'(t)\,\mathrm{d}t^2 = \frac{1}{\varepsilon^2}\mathrm{d}t^2. \tag{2.20}$$

This condition determines $\rho(t)$ in terms of $f(t)$ and this relation is quite simple close to the conformal boundary of AdS since $\rho \sim -\log\varepsilon - \log f'(t)$. This justifies dropping the first term above.

The conclusion of the previous paragraph is that the boundary degree of freedom is parametrized by a single function $f(t) \in \text{Diff}(S^1)$, but this space is clearly too large. Two boundary curves related by the isometries of AdS_2 should be considered equivalent even if they correspond to different profiles $f(t)$. To find this identification, it is simpler to go to the Poincare patch coordinates, and use the fact that since we are close to the conformal boundary $Z \sim 0$. The result is

$$F \to \frac{aF+b}{cF+d}, \qquad F = \frac{\beta}{\pi}\tan\frac{\pi f}{\beta}, \qquad ad - bc = 1. \tag{2.21}$$

Therefore the space of physically distinct boundary curves is $\text{Diff}(S^1)/\text{PSL}(2,\mathbb{R})$.

We can now evaluate the JT action on such configurations. The topological term gives $-S_0$ since $\chi = 1$ for the disk. The bulk dynamical term in JT gravity vanishes since locally

$R=-2$ everywhere. Finally the boundary term gives

$$K = 1 + \varepsilon^2 \Big\{ \tan \frac{\pi f(t)}{\beta}, t \Big\} + \mathcal{O}(\varepsilon). \tag{2.22}$$

In the action this multiplies $\sqrt{h}\Phi = \Phi_r/\varepsilon^2$. The '1' is therefore divergent while the second term is finite in the vanishing cutoff limit. We can easily remove the divergence by a boundary local counterterm $\oint \sqrt{h}\Phi$ which does not affect the variational problem. The action for the boundary curve (with matter sources turned off) is the Schwarzian theory

$$-I[f] = \underbrace{S_0}_{\text{from topological term}} + \underbrace{\Phi_r \int \mathrm{d}t \left\{ \tan \frac{\pi f(t)}{\beta}, t \right\}}_{\text{from dynamical terms}} \tag{2.23}$$

To write the action we lifted the element of $\text{Diff}(S^1)/\text{PSL}(2,\mathbb{R})$ to $f(t) \in \text{Diff}(S^1)$ and we consistently obtained an action which is invariant under $\text{PSL}(2,\mathbb{R})$.

The equation of motion of the Schwarzian action is $\frac{\mathrm{d}}{\mathrm{d}t}\{\tan \pi f/\beta, t\} = 0$. Up to a conformal transformation the solution is simply $f(t) = t$, a circle. This leads to a classical partition function

$$\log Z \sim S_0 + \frac{2\pi^2 \Phi_r}{\beta}, \qquad \rho(E) \sim e^{S_0}\, e^{2\pi\sqrt{2\Phi_r E}}. \tag{2.24}$$

We need to break the conformal symmetry to get a free energy that is reasonable, since otherwise we could only have either $\delta(E)$, a theory of ground states, or $1/E$, a continuous spectrum.

On-shell action

Compute the on-shell action of JT gravity on the hyperbolic disk with the NAdS_2 boundary conditions without using the Schwarzian action. Instead solve the equation of motion for the dilaton and evaluate the action directly and check eqn. (2.24).

What happened with the conformal symmetry? Time-reparemetrizations are broken by the Schwarzian action. Moreover, global conformal transformations acting on t are also broken. If Φ_r is zero, the symmetry is unbroken but fluctuations in the boundary shape are unsuppressed [15–18]. Equivalently, the low temperature limit of JT gravity is strongly coupled since the dimensionless coupling is the temperature itself in units of Φ_r.

2.5 Schwarzian theory: Partition Function

Let us recap what we did in path integral language. First integrate out the dilaton, localizing to hyperbolic metrics. For the topology of the disk there is only one choice. The integral over metrics reduces to a choice of boundary curve. Therefore the path integral reduces to a path integral over the Schwarzian mode:

$$Z(\beta) = e^{S_0} \int [\mathrm{d}g][\mathrm{d}\Phi]\; e^{\frac{1}{2}\int_M \mathrm{d}^2x\sqrt{g}\Phi(R+2) + \oint_{\partial M}\sqrt{h}\Phi(K-1)} \tag{2.25}$$

$$= e^{S_0} \int [\mathrm{d}f]\; e^{\Phi_r \int_0^\beta \mathrm{d}\tau \{\tan \frac{\pi f(\tau)}{\beta}, \tau\}}. \tag{2.26}$$

The measure on the first or second line can be derived from the BF analysis. Write the metric in the first-order variables and evaluate the symplectic form

$$\Omega = 2\int \mathrm{Tr}\Big[\delta_1 A \wedge \delta_2 A\Big]. \tag{2.27}$$

From this symplectic form one can derive a measure over the Schwarzian mode. This measure is precisely the natural measure over the coadjoint orbit $\mathrm{Diff}(S^1)/\mathrm{PSL}(2,\mathbb{R})$ derived in [24, 25].

Multiple ways have been developed to perform the Schwarzian path integral exactly. Regarding the partition function, we can use the Duistermaat-Heckman theorem as proposed by Stanford and Witten [26]. This is applicable since the integration space is symplectic and the Schwarzian derivative actually generates, via Poisson brackets, a U(1) symmetry that corresponds to time translations. This theorem implies two things 1) that the Schwarzian path integral is one-loop exact around fixed-points of the U(1) symmetry; and 2) that the one-loop determinant is equal to the product of "rotation angles". The output of the Gaussian integral is $1/\sqrt{\det' D}$, where D is the operator that generates the rotation symmetry of the fixed point, and the notation $\det'$ means that modes of D that can be generated by symmetries of the disk should be discarded. We note that the modes that are discarded are zero-modes in the sense that they do not appear in the action or in the symplectic form, but they are in general not zero-modes of D. So in general a few eigenvalues of D have to be omitted by hand.

In our case $D = \partial_t - 2\pi/\beta\partial_f$ the second term meaning we need to shift f so that $f(t) = t$ is indeed a fixed point. A Fourier mode expansion of $\delta f = f - t$ leads to

$$Z(\beta) \;=\; e^{S_0 + \frac{2\pi^2\Phi_r}{\beta}} \prod_{n\geq 2} \frac{\beta}{\Phi_r n} \;=\; \frac{\Phi_r^{3/2}}{4\sqrt{\pi}\beta^{3/2}} e^{S_0+\frac{\pi^2}{\beta}}. \tag{2.28}$$

It is clear that the rotation angle is proportional to $\Phi_r n/\beta$, while the fact that the prefactor is one is a consequence of a choice of normalization of the symplectic form. This normalization is arbitrary until we include higher topologies, which we will do in the next lecture. From now on we will set $\Phi_r = 1/2$ for simplicity. The linealization of the isometries leads to $\delta f \sim e^{\mathrm{i}2\pi n t/\beta}$ with $n = -1, 0, 1$ and hence the restricted range over n. The infinite product was taken via zeta-function regularization.

We can interpret (2.28) in holography. The partition function of such a system with Hilbert space $\mathcal{H}_{BH}$ and Hamiltonian H would be

$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}_{\rm BH}}\, e^{-\beta H} = \int dE\ \rho(E)\ e^{-\beta E},\ \ \rho(E) = \sum_n \delta(E - E_n), \tag{2.29}$$

where E_n is a discrete set of states of the quantum system describing the black hole. We can the exact Schwarzian partition function to infer what the density of states of this quantum system should be. An inverse Laplace transform of Eq. (2.28) gives

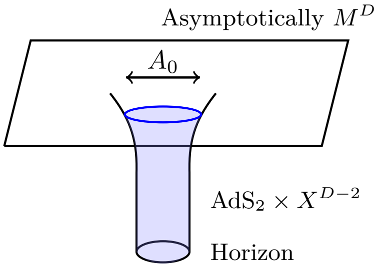
$$\rho_{\rm JT}(E) = \frac{e^{S_0}}{4\pi^2} \sinh\left(2\pi\sqrt{E}\right). \tag{2.30}$$

This result is nonperturbative in Φ_r, which suppresses perturbative metric fluctuations, but leading order in S_0.

Surprisingly, JT gravity on the disk with matter is also exactly solvable, even though correlators are no one-loop exact and no localization applies. The path integral with matter can be derived using the relation between $\mathrm{Diff}(S^1)/\mathrm{PSL}(2,\mathbb{R})$, representation theory of Virasoro, and Liouville CFT. This was the approach of Mertens, GJT, and Verlinde [27]. The results are reviewed in section 3 of [1].

2.6 JT gravity and near-extremal black holes

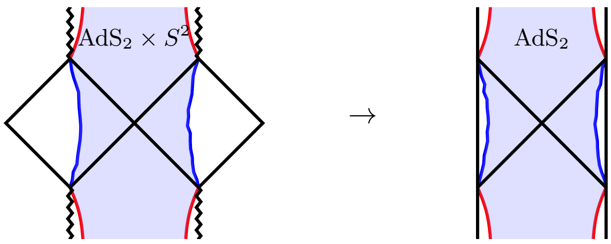
Besides being a toy model of quantum gravity and its relevance to the SYK model, JT gravity also describes the dynamics of certain higher-dimensional geometries. Near-extremal black holes universally have an $\mathrm{AdS}_2 \times X^{D-2}$ throat with an emergent isometry that includes the 1d conformal group. We can consider this in an asymptotically M^D space which might be AdS or flat.



(2.31)

It is useful to study the dynamics separately in the throat, and in the far-away region, and glue. Old idea implemented in multiple examples, recently [28–31]. For example:

Reissner-Nordstrom:



(2.32)

The gluing to the asymptotically flat region (which breaks the conformal symmetry) selects the Rindler patch time as the physical one. Furthermore, asymptotic observable determines boundary condition at the throat.

Reissner-Nordstrom

Derive the metric near the horizon and identify the presence of an AdS_2 factor.

E.g. the entropy arises mainly from near-horizon; Hawking radiation spectrum is determined by AdS_2 boundary two-point function. Higher-dimensional gravity in the throat is equivalent to JT gravity coupled to matter. Some comments:

JT-gravity sector: S_0 is the extremal Bekenstein-Hawking entropy. The JT metric $g_{\mu\nu}$ arises from 4d metric along temporal and radial directions. The dilaton Φ measures deviation of Area(X) from extremal value. Near the horizon $|\Phi| \ll S_0$, implying non-linear dilaton-potential terms suppressed by powers of $1/S_0$.

Matter sector: Arises from all other KK modes of higher-dimensional metric and matter. The radius of curvature of AdS and X area comparable, implying a large number of light matter fields. 2d gauge fields arise from higher-D gauge fields or isometries of X. Near the horizon $|\Phi| \ll S_0$, which implies that matter and dilaton interactions are suppressed by powers of $1/S_0$.

Boundary Condition: By including the main corrections from AdS_2 induced by the gluing to the far-away region one can derive the NAdS_2 boundary conditions (2.17) and extract Φ_r, see for example [32].

The quantum effects from the Schwarzian theory that we discussed above resolved some long-standing puzzles regarding black hole theormodynamics [33–35], as shown in [5–7]. This would be a topic of a separate set of lectures, for a quick summary see [36].

3 Lecture 2: Sum over topologies

3.1 Motivation

We have seen that JT gravity has two couplings: Φ_r, suppressing perturbative fluctuations, and S_0, suppressing topology. This is again something special about two dimensions since in higher dimensions both roles are played by G_N. We have solved the theory exactly in the first coupling. What about the second one?

A problem we encounter in the exact solution is a continuum spectrum. This connects with the information paradox as stated by Maldacena [37] which we now recall. Take the two-point function of some matter field and evaluate it at late times

$$\frac{1}{Z}\text{Tr}\left[e^{-\beta H}O(t)O(0)\right] = \sum_{n,m} e^{-\beta E_n} e^{-\text{i}t(E_m - E_n)} |\langle n|O|m\rangle|^2 \sim \text{Order one.} \tag{3.1}$$

Problem: If we compute this in gravity it decays exponentially with time for arbitrarily late times.

This is a problem because for a discrete spectrum the late time behavior shouldn't be too small. On average the dominant contribution should come from configurations with $E_n = E_m$ such that

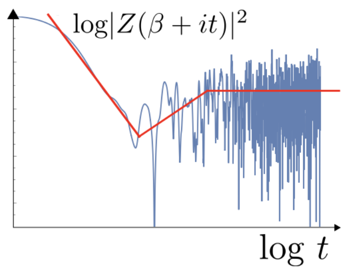
$$\frac{1}{Z}\text{Tr}\left[e^{-\beta H}O(t)O(0)\right] \sim \sum_{n} e^{-\beta E_n} |\langle n|O|n\rangle|^2 \sim \text{Order } e^{-S_0}. \tag{3.2}$$

Therefore the problem is correlated to the expectation that the black hole quantum system is finite dimensional so that $e^{S_0} < \infty$. In fact all black hole paradoxes are cored in the tension between a discrete spectrum and gravity. For example, there would be no issue with the entropy of Hawking radiation growing forever if the entropy of the black hole system was infinite.

It is useful to simplify the problem even futher. We can define an observable that is independent of a choice of operator and moreover is applicable even in pure gravity (where there is no such notion of a prefer operator in the bulk other than the Hamiltonian itself). This is the **spectral form factor**

$$\mathrm{SFF}(t) = \sum_{n,m} e^{-(\beta+\mathrm{i}t)E_n} e^{-(\beta-\mathrm{i}t)E_m}, \tag{3.3}$$

so this is clearly a product of partition functions $Z(\beta_1)Z(\beta_2)$ with $\beta_{1/2} = \beta \pm it$. The quantity starts off at $t = 0$ as order e^{2S_0} and oscillates erratically around the late-time mean $Z(2\beta)$ which is order e^{S_0} and therefore suppressed by a factor of e^{-S_0} with respect to early times. Its shape is roughly as (figure taken from [1])



The blue line is a member drawn from the GUE ensemble while the red line arises from averaging over Hamiltonians. One could also average over time windows. The ramp in the curve is characteristic of level repulsion.

Saad, Shenker and Stanford [38] propose to study pure JT gravity. They show that in order to see any sign of discreteness of the spectrum it is necessary to include spacetime wormholes, working at finite S_0. In particular they show pure JT gravity is equal to an average over quantum mechanical theories with a discrete spectra

$$Z_{\text{gravity}}(\beta_1, \ldots, \beta_n) = \int \mathrm{d}H\, P(H)\, \mathrm{Tr}\, e^{-\beta_1 H} \ldots \mathrm{Tr}\, e^{-\beta_n H}. \tag{3.4}$$

So pure gravity captures the average part of the spectral form factor. The precise ensemble of theories will involve a matrix potential and a double-scaling limit such that the spectral curve, defined through $y(x \pm \mathrm{i}\epsilon) = \mp \mathrm{i}\pi e^{-S_0}\rho_{\text{disk}}(x)$, is

$$\rho_{\text{disk}}(E) = \frac{e^{S_0}}{4\pi^2}\sinh(2\pi\sqrt{E}), \quad \Rightarrow \quad y(x) = \frac{1}{4\pi}\sin(2\pi\sqrt{-x}). \tag{3.5}$$

You learned about matrix models in week 1 and I assume more details on the double-scaling limit will be explained in C. Johnson's lectures.

This does not mean that we propose all holographic duals to involve disorder. We do expect the dual to have a chaotic spectrum. This is defined as a spectrum which shares statistical features with a random matrix, without necessarily being one. JT gravity is then simply a toy model that isolates the features of gravity responsible for chaos, namely spacetime wormholes.

3.2 Two-boundary wormhole

First consider the two-boundary wormhole with $g = 0$ and $n = 2$. This will be part of the building block for the general answer later.

The JT gravity path integral localizes to an integral over moduli space of hyperbolic surfaces with no handles and two boundaries. If we ignore boundary modes (which will be incorporated later) it seems the only moduli is the length of the interior geodesic, together with a twist.

How can we describe the moduli from the point of view of BF theory? A flat connection can be described by its holonomies around its non-trivial cycles. In this case we have only one which we can choose to be the interior geodesic. We denote the holonomy by $U \in G$. Two holonomies related by conjugation $U \to RUR^{-1}$ with $R \in G$ are considered gauge-equivalent so we only care about the conjugacy class.

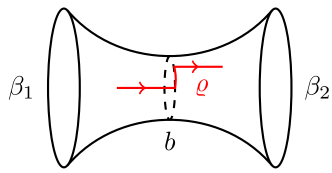
For a flat connection to be associated with a geometry, the holonomy should be hyperbolic. Any hyperbolic element U can be conjugated to

$$U = \pm \begin{pmatrix} e^{b/2} & 0 \\ 0 & e^{-b/2} \end{pmatrix}, \tag{3.6}$$

where b is the length of the geodesic. If we work with bosonic JT gravity, the overall sign can be discarded since we are working with $\mathrm{PSL}(2, \mathbb{R})$. In the presence of a spin structure, the overall sign indicates whether fermions are antiperiodic (NS) or periodic (R) around such cycle. This will be important in later lectures.

However, the length of the geodesic is not the only moduli. In the BF perspective on JT gravity, gauge transformations are constrained to be trivial along the boundaries. We can define a gauge-invariant "holonomy" V by parallel transport from one boundary to the other. One can show that the following combination $VUV^{-1}U^{-1}$ represents the holonomy of a contractible cycle which should therefore be trivial. This implies that V must commute with U when the connection is flat, so they are diagonal on the same basis $V = \pm\mathrm{diag}(e^{\varrho/2}, e^{-\varrho/2})$[2]. To avoid overcounting we restrict $0 \le \varrho \le b$. The interior moduli are therefore the length b and twist ϱ.

What is the geometric meaning of the twist? We can separate the hyperbolic cylinder into two trumpets bounded internally by the geodesic of length b. However there is a new moduli that arise from the gluing because we can act with a global rotation before gluing. This gluing is precisely represented in the BF description through V.



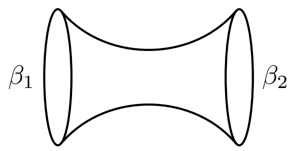
(3.7)

[2]What is the interpretation of the overall sign in a theory with fermions? Before we glue the two trumpets we can change the sign of fermions, producing a non-equivalent spin structure. For a given choice of NS/R boundaires, the cylinder will therefore have two choices of internal spin structure.

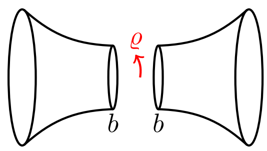
This picture hopefully clarifies why $\varrho \sim \varrho + b$. This is perhaps the simplest instance where the moduli of flat connections in the hyperbolic component (which would not put any constraint on ϱ) is distinguished from moduli space of hyperbolic surfaces.

The symplectic measure over length and twist is $\mathrm{d}b\,\mathrm{d}\varrho$, as we will see later. Since the path integral over the trumpets naturally do not depend on the twist parameter which is inherently associated to the gluing, we can integrate-out ϱ from the beginning, leading to an effective measure $b\,\mathrm{d}b$ over geodesic lengths.

Having described the interior moduli, we need to evaluate the path integral over the boundary wiggles described by the Schwarzian theory. We can then glue all contributions as shown in the figure



$$= \int_0^\infty \mathrm{d}b \int_0^b \mathrm{d}\varrho \tag{3.8}$$



The calculation is actually very similar to the one in the disk. The path integral localizes into hyperbolic surfaces and therefore there is no bulk contribution. Since the interior boundary is a geodesic $K = 0$ and the boundary term vanishes there. We end up with a theory very similar to the Schwarzian appearing on the disk but with a slightly different action that depends on b. The integration manifold is now

$$\mathrm{Diff}(S^1)/\mathrm{U}(1), \tag{3.9}$$

since the presence of the inner boundary breaks the group of isometries of the hyperbolic disk into only rotations. The rotation angles are nevertheless insensitive to the parameter b other than the fact that less modes are removed by isometries. We instead get

$$\prod_{n\geq 1} \frac{2\beta}{n} = \frac{1}{\sqrt{4\pi\beta}}. \tag{3.10}$$

Combining this with the value of the Schwarzian action evaluated on the fixed-point gives

$$Z_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta, b) = \frac{1}{\sqrt{4\pi\beta}}\, e^{-\frac{b^2}{4\beta}}. \tag{3.11}$$

Since the trumpet has the topology of an annulus, S_0 does not appear here.

We can now assemble the pieces and glue the contributions to the path integral from both trumpets together with the measure over interior moduli

$$Z_{0,2}(\beta_1, \beta_2) = \int_0^\infty b\,\mathrm{d}b\; Z_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta_1, b)\, Z_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta_1, b), \tag{3.12}$$

$$= \frac{1}{2\pi}\frac{\sqrt{\beta_1\beta_2}}{\beta_1 + \beta_2}. \tag{3.13}$$

Some comments on this result:

Match with matrix integral This is a very robust evidence in favor of the duality between JT gravity and random matrix models. In the double-scaling limit the leading order connected average of a product of two partition functions is universal (in a given symmetry class):

$$\langle \text{Tr}e^{-\beta_1 H}\text{Tr}e^{-\beta_2 H}\rangle^{\text{conn.}}_{g=0,n=2} = \frac{1}{2\pi}\frac{\sqrt{\beta_1\beta_2}}{\beta_1+\beta_2}\,e^{-\beta E_0}. \tag{3.14}$$

This matches our result in JT gravity since we are working in units where the threshold energy vanishes $E_0 = 0$.

The ramp We can analytically continue this result into complex $\beta_1 = \beta/2 + \mathrm{i}T$ and $\beta_2 = \beta/2 - \mathrm{i}T$, producing precisely the spectral form factor. The two-boundary wormhole will therefore produce precisely the ramp

$$\frac{1}{2\pi}\frac{\sqrt{\beta_1\beta_2}}{\beta_1+\beta_2} = \frac{1}{2\pi\beta}\sqrt{\frac{\beta^2}{4}+T^2} \to \frac{1}{2\pi}\frac{T}{\beta}. \tag{3.15}$$

Upon inverse Laplace transform this is a direct consequence of level repulsion, which in the density correlator leads to $\langle\rho(E_1)\rho(E_2)\rangle \sim -(E_1-E_2)^{-2}$.

Three-boundary wormhole We can easily extend this calculation to a surfaces with no handles and three boundaries. We can glue now three trumpets into a hyperbolic three-holed sphere. The simplification in this case arises because there is a single hyperbolic three-holed sphere with given boundary lengths b_1, b_2 and b_3. The gluing measure we derived applies independently to each of the three boundaries giving

$$Z_{0,3}(\beta_1,\beta_2,\beta_3) = \int_0^\infty \prod_{i=1,2,3} b_i\,\mathrm{d}b_i\, Z^{\text{trumpet}}_{\text{JT}}(\beta_i, b_i) = \frac{\sqrt{\beta_1\beta_2\beta_3}}{\pi^{3/2}}. \tag{3.16}$$

I'll leave it as an exercise to check this is the result predicted by the loop equations with $y(x) = \frac{1}{4\pi}\sin(2\pi\sqrt{-x})$.

3.3 Torsion

Surface with more boundaries or more handles will inevitably come with internal moduli that need to be integrated over. We can connect each boundary to a geodesic via a trumpet and therefore we can focus first on hyperbolic surfaces with geodesic boundaries. In this section we will explain how to obtain the measure using the torsion.

To define the path integral of a gauge theory one starts with a Riemannian metric on the fluctuation field δA such as $|\delta A|^2 = \int \text{Tr}\delta A \wedge \star \delta A$ which induces a Riemannian measure. This induces in turn a Riemannian measure on the space of zero-modes. We refer to this measure as μ_0. A one-loop calculation in the framework of the Fadeev-Popov procedure leads to the following quantum-corrected measure over the moduli space[3]

$$\mu = \mu_0 \cdot \frac{\det'\Delta_0}{\sqrt{\det'\Delta_1}} = \mu_0 \cdot \frac{\sqrt{\det'\Delta_0}}{\sqrt{\det'\Delta_2}}. \tag{3.17}$$

[3] Expand around a given flat connection A_0 and define $D = \mathrm{d} + [A_0, \cdot]$. This maps adjoint-valued q-forms to $(q+1)$-forms. The laplacian acting on adjoint-valued forms is $\Delta = D^*D + DD^*$.

The first equality makes it clear that the numerator comes from the ghosts and the denominator from the gauge field. The second identity arises from the Hodge decomposition of forms which implies that $\det'\Delta_1 = \det'\Delta_0 \det'\Delta_2$. On orientable manifolds we can define a Hodge star operator implying that $\det'\Delta_2 = \det'\Delta_0$ and therefore JT gravity is tree-level exact. This simlification is not available for unorientable manifolds.

The BF approach gives another description of this calculation. In any dimension, analytic torsion is a certain ratio of determinants times a classical measure μ_0 on the space of zero-modes [39, 40]. In two dimensions this reduces precisely to the measure μ above. The important result is that analytic torsion is equivalent to the combinatorial torsion of Reidemeister [41]. This is a quantity that can be evaluated on a lattice and the result is completely independent of the lattice. In the continuum limit the definition reduces to the analytic torsion but we can choose the simplest triangulation for evaluation. This was proven in [42–45].

In conclusion we have a few possible methods of calculation. The symplectic approach, which when dealing with bosonic JT gravity on orientable surfaces leads to the Weil-Petersson measure. This has the drawback of not being applicable for unorientable surfaces or not being practical for more complicated generalizations such as supersymmetry. The torsion is instead relatively easy to compute, and has the advantage of being straightforward to generalize to unorientable surfaces as well as supersymmetry.

The geodesic boundary condition of the trumpet is not obviously related to the boundary condition implicit in the torsion calculation. There is a subtlety in the statement that the symplectic structure and the torsion define the same measure. On an oriented two-manifold Σ without boundary, this is true as stated: if $\mathcal{T}_g$ is the moduli space of flat connections with gauge group G on a closed surface of genus g, then the symplectic structure and the torsion define the same measure μ on $\mathcal{T}_g$ (assuming the symplectic structure is properly normalized). On the other hand, if Σ is an oriented two-manifold of genus g with n boundary circles, then the symplectic structure and the torsion define measures on two closely related but slightly different spaces. The symplectic structure determines a measure μ on what we will call $\mathcal{T}_{g,n}$ (or $\mathcal{T}_{g,n}(\vec{w})$ if we wish to be more precise), the moduli space of flat bundles with prescribed conjugacy classes $\vec{w} = (w_1, \cdots, w_n)$ of the holonomies around the boundaries. The torsion defines a measure τ on what we will call $\mathcal{R}_{g,n}$, the moduli space of flat bundles over Σ without a restriction on the boundary holonomies. We will determine the relation between these two objects after evaluating the torsion.

3.3.1 The combinatorial torsion

The combinatorial torsion can be thought of as a framework dual to that of adjoint-valued forms. Consider a triangulation of the surfaces with q-dimensional cells and the boundary operator ∂ mapping q-cells to $q-1$-cells.

Instead of adjoint-value forms, we associate to each cell a vector space consisting of the adjoint representation of the group (this can be thought of as a covariantly constant section of the associated bundle E to the flat G-bundle). The boundary operator also acts by restricting the vector to its value at each boundary component.

The formal definition of the Reidemeister or combinatorial torsion is

$$\tau = \frac{\sqrt{\det' \partial_2^\dagger \partial_2}}{\sqrt{\det' \partial_1 \partial_1^\dagger}}. \tag{3.18}$$

This object is independent of triangulation and was shown to reduce to the analytic torsion in the continuum limit.

One can obtain a more useful version of this formula. For simplicity assume $H_2(Y, E) = 0$ (these are the homology groups associated to ∂). The torsion is

$$\tau = \frac{\alpha_2(s_1, \dots, s_{n_2})\alpha_0(\partial t_1, \dots, \partial t_{n_1-n_2-r}, v_1, \dots, v_k)}{\alpha_1(\partial s_1, \dots, \partial s_{n_2}, u_1, \dots, u_r, t_1, \dots, t_{n_1-n_2-r})}. \tag{3.19}$$

Let us define the objects involved in this formula:

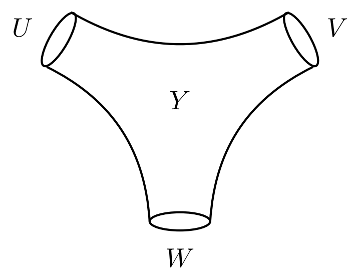
- $\alpha_q(v_1, \dots, v_{n_q})$ represents the measure of integration over the vector space living on the q-cells, when there are n_q of them. Since these are copies of the adjoint representation, a measure on the Lie group will naturally induce a measure.
- $s_1, \dots, s_{n_2}$ is a basis of the vector spaces of 2-cells. We assume $H_2(Y, E) = 0$ so the set of ∂s_j are linearly independent.
- $u_1, \dots, u_r$ where $r = \dim H_1(Y, E)$. $H_1(Y, E)$ is the cotangent bundle to the moduli space of flat bundles at the point E. Therefore r is the dimension of the moduli space of flat connections. The set $\{\partial s_1, \dots \partial s_{n_2}, u_1, \dots, u_r\}$ form a basis of 1-cells annihilated by ∂.
- $t_1, \dots, t_{n_1-n_2-r}$ elements that complete a basis of vector spaces of 1-cells.
- $v_1, \dots, v_k$ are extra basis vectors of the vector space of 0-cells in case $H_0(Y, E) \neq 0$.

The torsion is independent of the choice of basis $\{s_j\}$ and $\{t_j\}$. It does depend on the choice of $\{u_j\}$. This is fine since the result is a measure on $H^1(Y, E)$, the tangent space to $\mathcal{M}$.

3.3.2 The measure for JT gravity

Any surface can be decomposed into pairs of pants. Let us begin by analyzing this surface.

What is the moduli space of a sphere with three geodesic holes? Since there are three boundaries, flat connections will be specified by the holonomy along them which we can call U, V and W:



(3.20)

Not all these matrices are independent. As the figure makes it clear, a cycle made of the three boundaries simultaneously would be contractible. This implies that the holonomies ought to satisfy the constraint

$$UVW = 1. \tag{3.21}$$

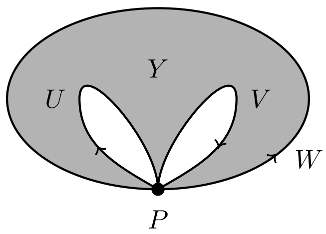
Finally, we should also mod out by an overall conjugation by a group element

$$(U, V, W) \cong (RUR^{-1}, RVR^{-1}, RWR^{-1}). \tag{3.22}$$

The constraint is evidently consistent with this identification.

The three matrices gives a total of 9 parameters. The constraint $UVW = 1$ provides 3 conditions, and moding out by an overall conjugation removes 3 parameters. This leaves a total of 3 parameters describing the moduli space of the three-holed sphere. These three parameters can be identified with the three geodesic lengths of the boundaries encoded in the conjugacy classes of U, V and W.

To compute the combinatorial torsion we can pick the simplest triangulation of the surface which is



(3.23)

Let us discuss the situation with an arbitrary gauge group G. Assume we divide the space of flat connections on Y only by gauge transformations that are trivial at P. Then the moduli space becomes simply $\widehat{\mathcal{R}} = G \times G$, parametrized by say U and V. The definition of the torsion requires a choice of left- and right-invariant measure vol_G on the G manifold. The formal definition of the combinatorial torsion gives the most naive answer one could have written down

$$\widehat{\tau} = \mathrm{vol}(U) \cdot \mathrm{vol}(V), \tag{3.24}$$

and

$$\tau = \frac{\mathrm{vol}(U) \cdot \mathrm{vol}(V)}{\mathrm{vol}(R)}. \tag{3.25}$$

The reason is elementary. Considering $\widehat{\tau}$ we effectively remove the point P from consideration. The contribution to the torsion from the vector space associated to the 2-cell can be shown to cancel with the one coming from any one of the three 1-cell which we take to be W[4]. The remainder is just the natural measure on U and V.

At this point the calculation becomes straightforward; one needs to find a convenient parametrization of the moduli space and compute these group measures. We can use the

[4]To write it more precisely the torsion is $\widehat{\tau} = \alpha_0(s_1, s_2, s_3)/\alpha_1(\partial s_1, \partial s_2, \partial s_3, u_1, u_2, u_3, v_1, v_2, v_3)$, where $\{u\}$ is a basis for vector space at U and $\{v\}$ a basis for vector space at V. One can show the contribution from s_1, s_2, s_3 cancel with $\partial s_1, \partial s_2, \partial s_3$ leaving only $\widehat{\tau} = \alpha_1^{-1}(u_1, u_2, u_3)\alpha_1^{-1}(v_1, v_2, v_3) = \mathrm{vol}(U) \cdot \mathrm{vol}(V)$.

freedom under conjugation to write $U = RU_0R^{-1}$ and $V = RV_0R^{-1}$ with

$$U_0 = \delta_1 \begin{pmatrix} e^{b_1/2} & \kappa \\ 0 & e^{-b_1/2} \end{pmatrix}, \quad V_0 = \delta_2 \begin{pmatrix} e^{b_2/2} & 0 \\ 1 & e^{-b_2/2} \end{pmatrix}. \tag{3.26}$$

($\delta = \pm 1$ which we can discard now but will be important later when we include spin structures.) This choice depends on three parameters: b_1 and b_2 which can obviously be interpreted as geodesics lengths, and κ. The latter parameter should be related to b_3, the geodesic length of the boundary with holonomy W. To determine the relation write $W = RW_0R^{-1}$ and

$$W_0 = V_0^{-1}U_0^{-1} = \delta_1\delta_2 \begin{pmatrix} e^{-(b_1-b_2)/2} & -\kappa e^{b_2/2} \\ -e^{-b_1/2} & e^{(b_1-b_2)/2} + \kappa \end{pmatrix}, \tag{3.27}$$

Compare the trace of W with that of a diagonal matrix,

$$\mathrm{Tr}W_0 = \delta_1\delta_2\Big(\kappa + 2\cosh\frac{b_1-b_2}{2}\Big) = \delta_3\, 2\cosh\frac{b_3}{2}, \tag{3.28}$$

combined with the fact that a spin structure is consistent if $\delta_1\delta_2\delta_3 = -1$. Again, we can forget about this here but it will be useful to keep in mind for the generalizations. This leads to

$$\kappa = -2\cosh\frac{b_3}{2} - 2\cosh\frac{b_1-b_2}{2}, \tag{3.29}$$

which is the relation we needed.

Finally we need to decide on a group measure. We can represent an element in the algebra $\mathfrak{sl}(2,\mathbb{R})$ by a 2×2 matrix $x = \begin{pmatrix} x_h & x_e \\ x_f & -x_h \end{pmatrix} = x_e\, e + x_f\, f + x_h\, h$. The measure derived from the inner product $|x|^2 = 2\mathrm{Tr}\, x^2$ is just $4\mathrm{d}x_e\mathrm{d}x_f\mathrm{d}x_h$. On a group element U we can write this measure as

$$\mathrm{vol}(U) = 4(U^{-1}\mathrm{d}U)_e(U^{-1}\mathrm{d}U)_f(U^{-1}\mathrm{d}U)_h. \tag{3.30}$$

Now we have all the ingredients we need to evaluate the torsion of the three-holed sphere which we leave as an exercise.

Torsion of the three-holed sphere

With this parametrization of U and V, and the definition of the measure, show that

$$\tau = 8\sinh\frac{b_1}{2}\sinh\frac{b_2}{2}\,\mathrm{d}b_1\,\mathrm{d}b_2\,\mathrm{d}\kappa \tag{3.31}$$

Using the relation between κ and b_3 we can rewrite the torsion as:

$$\tau = 8\sinh\frac{b_1}{2}\sinh\frac{b_2}{2}\sinh\frac{b_3}{2}\,\mathrm{d}b_1\,\mathrm{d}b_2\,\mathrm{d}b_3 \tag{3.32}$$

Fortunately we find that the final answer for the torsion turns up to be symmetric on the three boundaries. This is not manifest in our approach since we singled out U and V to parametrize the moduli space.

The prefactor looks funny. We expect the measure over the three-holed sphere to be independent of the boundary geodesic lengths. To answer this question we need to consider how the pair-of-pants are supposed to be glued to each other.

3.3.3 Gluing and torsion of a circle

What happens when we glue together two manifolds Y_1 and Y_2 with a common boundary S_{12}?

In QFT gluing is implemented by multiplying the path integrals over Y_1 and Y_2 and summing over physical states propagating along the common boundary S_{12}.

In the approach via the torsion to BF theory, the appropriate gluing procedure is to multiply the torsions of Y_1 and Y_2 and divide by the torsion of the circle S_{12}:

$$\frac{\tau_{Y_1} \cdot \tau_{Y_2}}{\tau_{S_{12}}}. \tag{3.33}$$

A detailed explanation of the procedure can be found in section 4 of [46].

Rough idea: We need to divide by the circle torsion to avoid overcounting. Otherwise gluing two pair-of-pants would lead to unwanted terms involving $(\mathrm{d}b)^2$. Let us now discuss the torsion of the circle.

Consider a flat connection on a circle with holonomy $U = \mathrm{diag}(e^{b/2}, e^{-b/2})$. This has an obvious 'triangulation', a base point in the circle, and the circle itself. The general procedure is slightly subtle and is outlined in section 3.4.3 of [47]. Denote by ∂ the linear transformation of the Lie algebra given by $\partial s = UsU^{-1} - s$. The definition of the torsion gives

$$\tau = \frac{\alpha_0(\partial t_1, \partial t_2, v)}{\alpha_1(\underbrace{t_1, t_2}_{\text{off-diagonal matrices}}, \underbrace{u}_{\text{diagonal matrix}})} = |\det{}' \partial| \, \frac{\alpha_0(v)}{\alpha_1(u)}. \tag{3.34}$$

The determinant over non-zero-modes leads after a simple calculation[5] to

$$|\det{}' \partial| = 4 \sinh^2 \frac{b}{2}. \tag{3.35}$$

What about $\alpha_0(v)/\alpha_1(u)$? The denominator arises from the tangent space to the moduli space of flat connections and should be associated to changes in the geodesic length through $U = \mathrm{diag}(e^{b/2}, e^{-b/2})$. The numerator arises from matrices in $\mathrm{SL}(2, \mathbb{R})$ that commute with U, which naturally appears when gluing through parallel transport across the circle and we called it $\mathrm{diag}(e^{\varrho/2}, e^{-\varrho/2})$. The ratio of measures is then naturally $\mathrm{d}b \, (\mathrm{d}\varrho)^{-1}$. The final answer is

$$\tau_S = 4 \sinh^2 \left(\frac{b}{2}\right) \mathrm{d}b \, \cdot \, (\mathrm{d}\varrho)^{-1} \tag{3.36}$$

[5] The eigenvalues of ∂ are $e^{\pm b} - 1$ and 0. The zero-mode, given by matrices that commute with U, is the generator u.

This is great, the factor of $\mathrm{d}b$ will cancel the extra unwanted term in the product of two three-holed-sphere torsion while the twist parameter $\mathrm{d}\varrho$ will replace it.

3.4 JT gravity as a matrix integral

Having determine the building blocks of the measure over hyperbolic surfaces relevant for JT gravity we can complete the calculation and show how the result can be reproduced by a matrix integral.

3.4.1 Measure of a closed surface

Let us begin with surfaces without boundaries. Any closed oriented surface Σ of genus g can be assembled by gluing together a set T of $2g-2$ three-holed spheres Y_t, $t \in T$. These three-holed spheres have to be glued along a set C of $3g-3$ circles S_c, $c \in C$. Two three-holed spheres (or two boundaries of the same three-holed sphere) are glued along each S_c.

$$\mu_g = \prod_{t\in T} \tau_{Y_t} \prod_{c\in C} \frac{1}{\tau_{S_c}}. \tag{3.37}$$

The result is quite simple. Each circle of length $\mathrm{d}b$ bounds two three-holed sphere whose torsion produces a factor $(\mathrm{d}b)^2$. The torsion of that circle replaces one $\mathrm{d}b$ by $\mathrm{d}\varrho$, leading a factor of $\mathrm{d}b\,\mathrm{d}\varrho$ for each circle. Moreover one can easily see that all factors of $2\sinh b/2$ nicely cancel between the three-holed spheres and the circles.

The final torsion of a closed surface of genus g and no boundaries $n = 0$ is given by

$$\mu_g = \prod_{i=1}^{3g-3} \mathrm{d}b_i\,\mathrm{d}\varrho_i \tag{3.38}$$

This is precisely the Weil-Petersson volume form you probably learned about last week! We could have derived this from a symplectic approach but, as we will see, the torsion is more powerful. (Since gluing is done locally, this should also be the measure gluing the two trumpets in the cylinder.)

The JT gravity partition function on closed hyperbolic surfaces is then given by

$$Z_{g,0} = \int_{\mathcal{M}_{g,0}} \prod_{i=1}^{3g-3} \mathrm{d}b_i\,\mathrm{d}\varrho_i = V_{g,0}, \tag{3.39}$$

where $V_{g,n=0}$ is the corresponds Weil-Petersson volume. The integral is done over $\mathcal{M}_{g,0}$, the moduli space of hyperbolic surfaces. This involves fixing the right component of the moduli space of flat connections, Teichmuller space $\mathcal{T}_{g,0}$, but also involves modding out by the mapping class group[6]; two surfaces with very different values of b_i and ϱ_i might be non-trivial equivalent under a large diffeomorphism. Fortunately, once we get to this point, we can use the results from Mirzakhani [48, 49] that you learned from Giacchetto and Lewanski's lectures.

[6]The measure we derived on $\mathcal{T}$ naturally descends to a measure on $\mathcal{M}$. The reason is that it was obtained from a gravity calculation and diffeomorphisms, small or large, is a symmetry of the theory.

The JT gravity partition function without boundaries is equal to the Weil-Petersson volumes, which combined with the result of [50] implies that they are computed by a double-scaled matrix integral! Precisely the spectral curve derive from the JT gravity disk partition function is the one identified by Eynard and Orantin as required to reproduce the Weil-Petersson volumes.

3.4.2 Final answer

To complete the calculation we need to incorporate boundaries. We would like to fix boundary conditions that are naturally in holography leading to the so-called NADS_2 boundaries we introduced in the previous lecture. This cannot be done in the language of the torsion, but can be inferred from previous results.

For example, while $Z_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta, b)$ is a number, the path integral with torsion boundary conditions

$$\widetilde{Z}_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta, b)\, \mathrm{d}b, \tag{3.40}$$

should be a measure of integration over b. To determine it we can compare the two-boundary wormhole

$$Z_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta_1, b)\ \mathrm{d}b\mathrm{d}\varrho\ Z_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta_2, b), \tag{3.41}$$

with the quantity with torsion boundary conditions

$$\widetilde{Z}_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta_1, b)\, \mathrm{d}b\, \frac{1}{\tau_S}\, \widetilde{Z}_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta_2, b)\, \mathrm{d}b. \tag{3.42}$$

Comparing both quantities we infer that the trumpet path integral with torsion boundary conditions is given by

$$\widetilde{Z}_{\mathrm{JT}}^{\mathrm{trumpet}}(b) = Z_{\mathrm{JT}}^{\mathrm{trumpet}} \cdot 2\sinh\frac{b}{2}. \tag{3.43}$$

We are ready to write the final answer for the JT gravity path integral with NAdS boundaries. First of all, the argument for the case without boundaries still implies that the integration over all internal geodesics is done with a measure $\prod_i \mathrm{d}b_i\, \mathrm{d}\varrho_i$. What about geodesic connected to trumpets? Now we have to worry about geodesics connected to trumpets. The torsion of the three-holed sphere, together with the trumpet and circle torsions, will include term

$$\prod_{i\in C_{\mathrm{int}}} \mathrm{d}b_i\, \mathrm{d}\varrho_i \cdot \prod_{e\in C_{\mathrm{ext}}} \underbrace{2\sinh\frac{b_e}{2}\mathrm{d}b_e}_{\text{Leftover from torsion of three-holed sphere}} \frac{1}{\tau_{S_e}} \underbrace{Z_{\mathrm{JT}}^{\mathrm{trumpet}} \cdot 2\sinh\frac{b_e}{2}\mathrm{d}b_e}_{\text{Trumpet with torsion bdy}} \tag{3.44}$$

where C_{int} is the set of internal geodesics while C_{ext} is the set of geodesics connected to external boundaries. The JT gravity partition function is the integral of the moduli space of hyperbolic surfaces using this measure. To relate it to the quantity computed by

Mirzakhani, we can write the final answer as[7]

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \prod_{e\in C_{\rm ext}} \int_0^\infty b_e\, {\rm d}b_e Z^{\rm trumpet}_{\rm JT}(\beta_e,b_e) \qquad \underbrace{\int_{\mathcal{M}_{g,\vec{b}}} \prod_i {\rm d}b_i\, {\rm d}\varrho_i}_{\text{Weil-Petersson volume } V_{g,n}(b_1,\ldots,b_n)} \quad . \tag{3.45}$$

The internal Weil-Petersson volume does not care about the twist parameter used for the external gluing and is therefore independent of it. This allows us to perform the integral over ϱ_e rather trivially and obtain factors of b_e.

The answer in its final form becomes, after writing explicitly the trumpet path integral,

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \left(\prod_{e=1}^n \int_0^\infty b_e {\rm d}b_e\, \frac{e^{-b^2/(4\beta)}}{\sqrt{4\pi\beta}}\right) V_{g,n}(b_1,\ldots,b_n). \tag{3.46}$$

Now all the ingredients are explicit and known. The Eynard-Orantin result now implies that

$$Z_{g,n} = \left\langle {\rm Tr}\, e^{-\beta_1 H} \ldots {\rm Tr}\, e^{-\beta_n H}\right\rangle^{\rm conn.}_{g,n} \tag{3.47}$$

JT gravity as a matrix integral

Complete the proof that $Z_{g,n}$ computed via JT gravity is the same as the matrix integral with an insertion of ${\rm Tr}\, e^{-\beta H}$ and the Schwarzian spectral curve. To do this first relate $Z(\beta)$ to the resolvent $R(E)$ via

$$R(E) = -\int_0^\infty {\rm d}\beta\, e^{\beta E}\, Z(\beta). \tag{3.48}$$

Then apply this result to the JT gravity answer, and show the result is the same as the Laplace transform relating $R_{g,n}(x_1,\ldots,x_n) \leftrightarrow V_{g,n}(b_1,\ldots,b_n)$ discovered by Eynard and Orantin.

4 Lecture 3: Generalizations

4.1 Dilaton-gravity as a matrix integral

Matrix integrals on a given symmetry class are parametrized by one function, the matrix potential, or equivalently the spectral curve. Two-dimensional dilaton-gravity theories are also parameterized by a function, the dilaton potential. Could the two theories be related beyond the specific example of JT gravity? If so, what is the relation between the spectral curve and the dilaton potential?

Under some assumptions on the dilaton potential, both questions were answered positively by Maxfield and GJT [51] and independently by Witten [52].

[7] We denote by $\mathcal{M}_{g,\vec{b}}$ the moduli space with geodesic boundaries to distinguish from $\mathcal{M}_{g,n}$, the moduli space with n punctures.

The 2d dilaton-gravity action in terms of the dilaton potential takes the form

$$I = -\frac{S_0}{4\pi}\int_M \sqrt{g}R - \frac{1}{2}\int_M \sqrt{g}(\Phi R + U(\Phi)) + I_{\rm bdy}. \tag{4.1}$$

It is convenient to consider dilaton potentials of the following form

$$U(\Phi) = 2\Phi + \sum_I \lambda_I \; e^{-(2\pi-\alpha_I)\Phi}. \tag{4.2}$$

What's special about this choice? Consider the theory with a single exponential for simplicity. Since we already solved the theory with $\lambda = 0$ it is reasonable to explore how a Taylor expansion in λ would look like. To each order in λ the result would look like the JT gravity partition function with an insertion of

$$\frac{\lambda^k}{k!}\,\sqrt{g}\,\int [{\rm d}g\,{\rm d}\Phi]\,\Big(\prod_{j=1}^{k}\int_M {\rm d}^2x_j\sqrt{g}\,e^{-(2\pi-\alpha)\Phi(x_j)}\Big)\,e^{-I_{JT}}\,. \tag{4.3}$$

Recall that the JT gravity action is linear in Φ which acts as a Lagrange multiplier setting $R = -2$. Imagine we can commute the two integrals, performing first the JT gravity path integral with an insertion of $\prod_j e^{-(2\pi-\alpha)\Phi(x_j)}$ with fixed positions. The effect of the exponential insertions become very simple, as observed earlier in [53], replacing the constant negative curvature condition by

$$R+2 = 2\sum_j (2\pi-\alpha)\,\delta(x_j). \tag{4.4}$$

Notice that in the end we are required to integrate over the positions. The prefactor of $1/k!$ just guarantees that defects are treated as indistinguishable. This means the path integral localizes into hyperbolic metrics with conical singularities. In this notation α denotes the opening angle of the defect so that the singularity vanishes when its $\alpha = 2\pi$. The overall factor of λ^k means that each defect insertion comes with a weight λ while the prefactor $1/k!$ imposes the fact that the defects are indistinguishable.

After this observation the procedure is clear. We should first determine the JT gravity path integral in the presence of a fixed number k of defects. Next we should sum over all possible values of k. It is surprising that the second step can be performed in closed form for the disk allowing us to extract the deformed spectral curve. Moreover, we will also show the result is still dual to a matrix integral with that density of states.

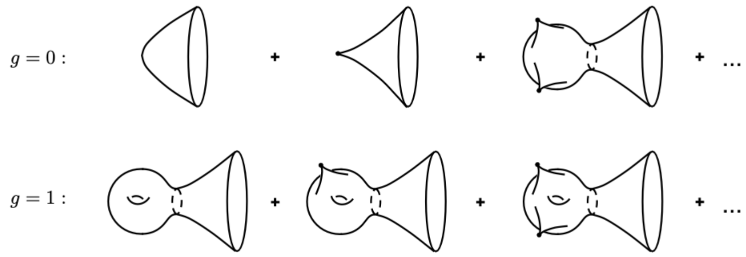
We can write the path integral of dilaton gravity as

$$Z(\beta_1,\ldots,\beta_n) = \sum_{g=0}^{\infty} e^{S_0\chi} Z_{g,n}(\beta_1,\ldots,\beta_n) \tag{4.5}$$

with

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\,Z_{g,n,k}(\beta_1,\ldots,\beta_n;\alpha_1,\ldots,\alpha_k). \tag{4.6}$$

In this case we consider a single defect species so all α's are the same but this can be generalized in a straightforward fashion. This can be represented in the figure, taken from [51],



Now we need to evaluate $Z_{g,n,k}$. Let us begin with the simplest case of the disk with a defect. In this case the theory reduces to the Schwarzian mode which now lives in the space

$$\mathrm{Diff}(S^1)/\mathrm{U}(1). \tag{4.7}$$

We can apply the Duistermaat-Heckman theorem again. We obtain

$$Z_{0,1,1}(\beta) = \frac{1}{\sqrt{4\pi\beta}} e^{\frac{\alpha^2}{4\beta}}. \tag{4.8}$$

Notice that this is same as the trumpet partition function under the replacement

$$b \to \mathrm{i}\alpha. \tag{4.9}$$

Disk with defect

Without using the Schwarzian theory, find the classical solution of JT gravity on a hyperbolic disk with a defect and evaluate its on-shell action. Show it is given by $\alpha^2/(4\beta)$.

This is not a coincidence. We can compare the geodesic hole to the defect using the BF formulation. The holonomy of a hole of length b and a conical singularity of opening α are

$$U_b = \delta \exp\begin{pmatrix} b/2 & 0 \\ 0 & -b/2 \end{pmatrix}, \quad U_\alpha = \delta \exp\begin{pmatrix} 0 & \alpha/2 \\ -\alpha/2 & 0 \end{pmatrix} \tag{4.10}$$

These two holonomies are conjugate, preserving the spin structure δ, if we identify $b = \mathrm{i}\alpha$. This is the simplest fact indicating that one might be able to obtain results for defects by analytic continuation on geodesic lengths of holes.

The relation between holes and defect continue to hold in more complicated surfaces if and only if all opening angles satisfy $0 \leq \alpha \leq \pi$. In fact, it was proven by Tan Wong and Zhang [54], and further developed by Do and Norbury [55], that the Weil-Petersson volumes of moduli spaces of hyperbolic surfaces with n holes of length $\vec{b}$ and k defects $\vec{\alpha}$ in this range are given by

$$V_{g,n,k}(\vec{b};\vec{\alpha}) = V_{g,n+k}(b_1,\ldots,b_n,\mathrm{i}\alpha_1,\ldots,\mathrm{i}\alpha_k). \tag{4.11}$$

When $0 \leq \alpha \leq \pi$ all boundaries are homologous to geodesics without encountering any defect. This directly implies that the partition function can be obtained by gluing

$$Z_{g,n,k}(\beta_1, \ldots, \beta_n) = \left(\prod_{e=1}^{n} \int_0^\infty b_e \, \mathrm{d}b_e \, Z_{\mathrm{JT}}^{\mathrm{tr}}(\beta_e, b_e) \right) V_{g,n,k}(\vec{b}; \vec{\alpha}) \tag{4.12}$$

We have not included the factor of $1/k!$. The reason is that the volumes are computed naturally for distinguishable defects.

The disk with defects: The behavior near the edge does not take the form of a matrix integral. The correction to the density of states from one defect gives

$$\rho_{k=1}(E) = \frac{\lambda e^{S_0}}{2\pi} \frac{\cosh(\alpha\sqrt{E})}{\sqrt{E}} \sim \frac{\lambda e^{S_0}}{\sqrt{E}}. \tag{4.13}$$

Does this mean the duality fails? No, actually contributions with more defects diverge faster, signaling that a resummation is necessary to decide what the actual behavior at the edge is.

We would like to evaluate the low temperature limit of the partition function. The Weil-Petersson volumes have the limit, when one of the boundaries is large,

$$V_{0,k+1}(b_0, \vec{b}) = \frac{1}{(k-2)!} \frac{b_0^{2k-4}}{2^{k-2}} + \ldots, \qquad b_0 \gg 1 \tag{4.14}$$

Volumes for large boundaries

Derive the generalization of this formula for arbitrary genus.

This implies that

$$Z_{0,1,k}(\beta) = \frac{1}{\sqrt{2\pi}} (2\beta)^{k-3/2} + \ldots, \tag{4.15}$$

which after summing of the defect leads to

$$Z(\beta) = e^{S_0} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} Z_{0,1,k}(\beta) = \frac{1}{4\sqrt{\pi}\beta^{3/2}} e^{2\lambda\beta} + \ldots. \tag{4.16}$$

In terms of the density of states

$$\rho(E) \sim e^{S_0} \frac{1}{2\pi} \sqrt{2(E - E_0)}, \quad E_0 = -2\lambda + O(\lambda^2). \tag{4.17}$$

Therefore the singularity at $E = 0$ is simply signaling the fact that the deformation includes a shift in the threshold energy. Another result of [51] was to point out the relevance of this fact to the question of whether pure 3d gravity exists.

Exact density of states: Using the following exact formula for Weil-Petersson volumes derived in [56][8]

$$V_{0,n}(b_1,\dots,b_n) = \frac{(-1)^{n-1}}{2}\frac{\mathrm{d}^{n-3}}{\mathrm{d}x^{n-3}}\left[u'(x)\prod_{i=1}^{n} J_0\left(b_i\sqrt{u(x)}\right)\right]\Big|_{x\to 0}, \quad \frac{\sqrt{u}}{2\pi}I_1(2\pi\sqrt{u}) = x \tag{4.18}$$

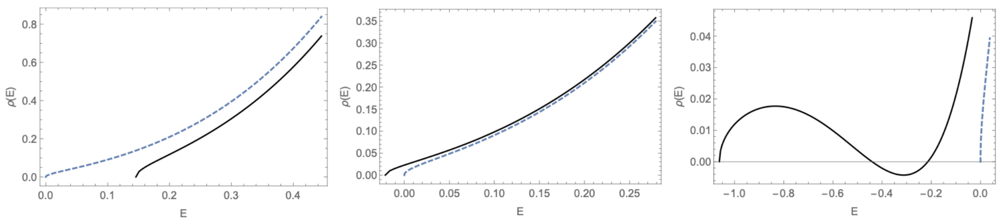
we can derive the exact density of states by summing over an arbitrary number of defects on the disk. The derivation is quite involved and can be found in section 3 of [51]. The final result is

$$\rho(E) = \frac{e^{S_0}}{2\pi}\int_{E_0}^{E}\frac{\mathrm{d}u}{\sqrt{E-u}}\,\frac{\mathrm{d}F(u)}{\mathrm{d}u}, \tag{4.19}$$

where

$$F(u) = \frac{\sqrt{u}}{2\pi}I_1(2\pi\sqrt{u}) + \sum_I \lambda_I\, I_0(\alpha_I\sqrt{u}). \tag{4.20}$$

The threshold energy is the largest root of $F(E_0) = 0$ and one can show the edge is a square root, as universally expected for a random matrix spectrum. This function $F(u)$ is precisely the tree-level string equation that appeared on C. Johnson's lectures. Even though the expression for $\rho(E)$ is quite complicated, its nice that the defect species are additive in the string equation. The shape of the density of states looks like (figure taken from [51])



(a) $\alpha = 1/2$, $\lambda = -0.1$ (b) $\alpha = 1/2$, $\lambda = 0.01$ (c) $\alpha = 1/2$, $\lambda = 0.08$

As a side comment, it is interesting that the density of states can become negative. This was further studied by Rosso and Johnson in [58]. In [59] we show that in some simpler settings, namely $\mathcal{N} = 1$ supergravity, the matrix model indicates a transition to a two-cut phase. Open question: What is the gravity interpretation of such phase transitions?

Proof of the duality To prove the duality at all orders in the genus expansion, it is useful to write the partition function in a slightly different way. First we remind that the resolvent and Weil-Petersson volumes are related by

$$W_{g,n}(\vec{z}) = \left(\prod_{e=1}^{n}\int_0^\infty b_e\,\mathrm{d}b_e e^{-b_e z_e}\right)V_{g,n}(\vec{b}). \tag{4.21}$$

[8] The derivation can be done in the language of the orthogonal polynomial method that C. Johnson described in his lecture. The equation for $u(x)$ is the tree-level string equation. The formula for $V_{g,n}$ arises from a general formula for genus-zero correlators for any double-scaled matrix integral derived in [57].

where $W_{g,n}(\vec{z}) = (-2z_1)\dots(-2z_n)R_{g,n}(-z_1^2,\dots,-z_n^2)$. We can write the deformed $W_{g,n}(\vec{z};\lambda)$ and evaluate the derivative with respect to λ at $\lambda = 0$. These are the path integrals with defects precisely. Some simple manipulations lead to

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} W_{g,n}(\vec{z})\Big|_{\lambda=0} = \left(\int_0^\infty \prod_{e=1}^{k} b_e\, \mathrm{d}b_e e^{-b_e z_e}\right) V_{g,n+k}(b_1,\dots,b_n,\underbrace{\mathrm{i}\alpha,\dots,\mathrm{i}\alpha}_{k\text{ terms}}) \tag{4.22}$$

We can combine this with an inverse Laplace transform

$$V_{g,n}(\vec{b}) = \left(\prod_{e=1}^{n}\int_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{e^{b_e z_e}}{b_e}\right) W_{g,n}(\vec{z}). \tag{4.23}$$

This leads to

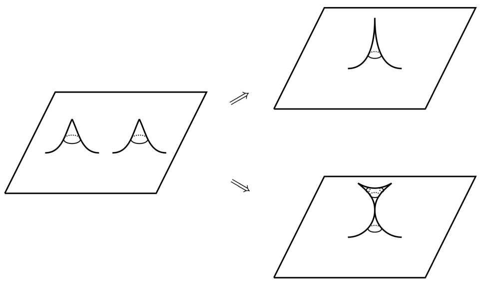
$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} W_{g,n}(\vec{z})\Big|_{\lambda=0} = \left(\prod_{e=1}^{k}\int_{\mathcal{C}} \frac{\mathrm{d}\tilde{z}_e}{2\pi\mathrm{i}}\frac{\sin(\alpha\tilde{z}_e)}{\alpha}\right) W_{g,n+k}(\vec{z},\tilde{z}_1,\dots,\tilde{z}_k), \tag{4.24}$$

where we write $e^{\mathrm{i}\alpha\tilde{z}_e}/\mathrm{i}\alpha$ in terms of the sine for convenience; it guarantees the formula is valid for $(g,n) = (0,1)$ as well. We can take as integration contour a curve that encircles the cut. Finally this is equivalent to the relation at finite λ

$$\frac{\mathrm{d}W_{g,n}(\vec{z};\lambda)}{\mathrm{d}\lambda} = \oint_{\mathcal{C}} \frac{\mathrm{d}\tilde{z}}{2\pi\mathrm{i}} f(\tilde{z})\, W_{g,n+k}(\vec{z},\tilde{z};\lambda), \quad f(z) = \frac{\sin(\alpha z)}{\alpha} \tag{4.25}$$

This is precisely of the form relevant for the "deformation theorem" of Eynard and Orantin [50], which states that deformations that take this form satisfy the topological recursion[9].

Blunt Defects When the defects are blunt the previous results are not valid. This is the range $\pi < \alpha \leq 2\pi$. The problem is essentially to take care of situations where defects can merge such as



The way the Deligne-Mumford compactification works, the merger of blunt defects is not correctly treated.

We might think the result can be simply analytically continued to this range but this is wrong. For example the Weil-Petersson volume on the sphere with four defects would be

$$V_{0,0,4}(\vec{\alpha}) =? \frac{4\pi^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 - \alpha_4^2}{2}, \tag{4.26}$$

[9] The theorem is even more general and allows the integrand, which in our case is $\sin(\alpha z)/z$ to also depend on λ. We didn't find an application of this generalization to gravity.

However, even when obeying the Gauss-Bonnet constraint

$$\frac{1}{2}\int R + \sum_j (2\pi - \alpha_j) = 2\pi\chi, \tag{4.27}$$

which implies that $\sum_j (2\pi - \alpha_j) \geq 4\pi$, one can obtain negative answer, for example for $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 > \sqrt{2}\pi$ ($\sum_j (2\pi - \alpha_j) = 8\pi - 2\sqrt{2}\pi = 2\pi(4 - \sqrt{2})$). Another evidence is that when we put $\alpha = 2\pi$, turning the defect into nothing, we do not recover the answer with one defect less!

What should the answer be instead? An answer was proposed in [60] based on considerations regarding the minimal string. The answer is given by a matrix integral with tree-level string equation

$$F(u) = \int_{\mathcal{C}} \frac{\mathrm{d}y}{2\pi\mathrm{i}}\, e^{2\pi y}\left(y - \sqrt{y^2 - u - 2W(y)}\right), \quad W = \sum_i \lambda_i e^{-(2\pi - \alpha_i)y}. \tag{4.28}$$

You can translate this to a density of states as in C. Johnson's lectures. In particular when $0 \leq \alpha \leq \pi$ it reduces to the string equation at tree-level written above. We verified some non-trivial limit of this formula in [60]. For example, when $\alpha = 2\pi$ we recover the JT gravity curve.

In [61] we showed that the Weil-Petersson measure on the moduli space with boundaries and defects is given by

$$\frac{\omega}{2\pi^2} = \kappa_1 + \sum_i \frac{b_i^2}{4\pi^2}\psi_i - \sum_j \frac{\alpha_j^2}{4\pi^2}\psi_i + \sum_{I \subset \{1,\ldots,m\}, \alpha_I \geq 0} \frac{\alpha_I^2}{4\pi^2}\delta_{0,I} \tag{4.29}$$

where the last sum is over subsets of the defects and

$$\alpha_I = 2\pi - \sum_{i \in I}(2\pi - \alpha_i). \tag{4.30}$$

Finally, $\delta_{g,I} \in \mathrm{H}^2(\mathcal{M}_{g,n})$ are the Poincare duals of boundary divisors, which in physics language can be interpreted as a contact term when vertex operators collide. The boundary divisor separates a surface of genus g with the punctures inside the set I.

Open question: Give a proof of this duality!

4.2 Fermionic JT gravity

Fermionic JT gravity can be defined, similarly to bosonic JT gravity, in terms of a BF theory with group $\mathrm{SL}(2,\mathbb{R})$. The sign that we dropped in the bosonic theory encodes the information about the surfaces spin structure.

Define the mod 2 index theory, considering closed surfaces first. Then we can study the Dirac operator $\not{D}$. Zero-modes come in pairs of opposite chirality. The number of say positive chirality zero-modes mod 2 is a topological invariant we call ζ. Then we can define a TQFT by $(-1)^\zeta$. A nice discussion can be found in section 3 of [62]. If one wants to define it on a surface with a boundary one runs into an anomaly which precisely reproduces something we will see below[10].

Without mod 2 index: The fermionic JT gravity partition function therefore

$$\sum_{\text{spin}} 1 = 2^{2g+n-2}(1+(-1)^{n_R}) \tag{4.31}$$

This is consistent with the fact that an orientable two-manifold with a spin structure has an even number of Ramond boundaries[11].

This can be reproduced by the following matrix integral. Consider a Hilbert space $\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f$ composed of bosonic and fermionic subspaces of dimensions L_b and L_f, respectively. Assume for simplicity that $L_b = L_f = L/2$ where $L = \dim\mathcal{H}$. Work in a basis where $(-1)^{\mathsf{F}}$ is represented as

$$(-1)^{\mathsf{F}} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right). \tag{4.32}$$

Consider a Hamiltonian of the form

$$H = \left(\begin{array}{c|c} H_b & 0 \\ \hline 0 & H_f \end{array}\right), \tag{4.33}$$

where H_b and H_f are statistically independent matrices drawn from the GUE ensemble. Moreover, a boundary with NS structure corresponds to an insertion of

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}_b}\, e^{-\beta H_b} + \mathrm{Tr}_{\mathcal{H}_f}\, e^{-\beta H_f}, \tag{4.34}$$

while a boundary with R structure corresponds to an insertion of

$$\mathrm{Tr}_{\mathcal{H}}\, (-1)^{\mathsf{F}}\, e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}_b}\, e^{-\beta H_b} - \mathrm{Tr}_{\mathcal{H}_f}\, e^{-\beta H_f}. \tag{4.35}$$

Consider H_b and H_f to be statistically independent. To make this as clear as possible I'll write all steps. The correlators in the large e^{S_0} limit are given by

$$\begin{aligned} \left\langle \prod_{i=1}^{n_{\mathrm{NS}}} Z_{\mathrm{NS}}(\beta_i) \prod_{i=1}^{n_{\mathrm{R}}} Z_{\mathrm{R}}(\beta_i) \right\rangle_{g,n}^{\text{conn.}} &= \left\langle \prod_{i=1}^{n} \mathrm{Tr}_{\mathcal{H}_b}\, e^{-\beta H_b} \right\rangle + (-1)^{n_{\mathrm{R}}} \left\langle \prod_{i=1}^{n} \mathrm{Tr}_{\mathcal{H}_f}\, e^{-\beta H_f} \right\rangle, \\ &= (1+(-1)^{n_{\mathrm{R}}}) \cdot (\text{GUE correlator, } e^{S_0} \to e^{S_0}/2), \\ &= \underbrace{2^{2g+n-2}(1+(-1)^{n_{\mathrm{R}}})}_{\text{Sum over spin structures}} \times \underbrace{(\text{GUE correlator, } e^{S_0})}_{\text{Reproduced by JT gravity amplitudes}} \end{aligned} \tag{4.36}$$

[10] A convenient way to compute the mod 2 index on surfaces with boundaries is by comparing different spin structures that are the same on $\partial\Sigma$. This is the relevant question anyways in the application to holography. The reason is that the mod 2 index is local so can remove the boundaries (glue them for example) and then evaluate it through Dirac operator zero-modes.

[11] In case you never thought about this, a simple way to see this is the following. We already noticed that a three-holed sphere satisfies $\delta_1\delta_2\delta_3 = -1$. The product of $\delta_1\delta_2\delta_3$ over all three-holed spheres gives $\prod_Y \delta_{Y_1}\delta_{Y_2}\delta_{Y_3} = (-1)^{\chi} = (-1)^n$. On the other hand since all internal boundaries are counted twice, since they are shared by two three-holed spheres, then $\prod_Y \delta_{Y_1}\delta_{Y_2}\delta_{Y_3} = (-1)^{n_{\mathrm{NS}}}$. Therefore $(-1)^{n_{\mathrm{NS}}} = (-1)^{n-n_{\mathrm{R}}} = (-1)^n$ implying that $(-1)^{n_{\mathrm{R}}} = 1$.

In the first line we used the statistical independence combined with the fact that $(-1)^{\mathsf{F}}$ will insert a minus sign for each Ramond boundary. In the second line we used the fact that both H_1 and H_2 have the same dimension and therefore both contribute the same as the GUE ensemble with a dimension given by half the total Hilbert space dimension. This rescaling produces a prefactor times the GUE correlator with conventional normalization of e^{S_0}. The prefactor is precisely $\sum_{\rm spin} 1$ while the GUE correlator is given by the bosonic JT gravity path integral, completing the duality.

With mod 2 index: Next, consider the theory that does include the mod 2 anomaly in the bulk. The result is now multiplied by

$$\sum_{\rm spin}(-1)^{\zeta} = 2^{g+n-1}\delta_{n_R,0}. \tag{4.37}$$

Notice that the power that appears here is not the Euler characteristic and therefore cannot be aboserved fully in a shift of S_0. (This is only true with boundaries! In closed surfaces $n=0$ and we can shift e^{S_0} making the theory trivial.)

To reproduce this answer we are forced to impose

$$\mathrm{Tr}\,(-1)^{\mathsf{F}} e^{-\beta H} = 0, \tag{4.38}$$

for each member of the ensemble (as opposed to on average, as in the previous case). Assume also that an insertion of a NS boundary corresponds to

$$\sqrt{2}\mathrm{Tr}e^{-\beta H} \tag{4.39}$$

Then

$$\underbrace{(\sqrt{2})^n(\sqrt{2})^{2g+n-2}}_{2^{g+n-1}=\sum_{\rm spin}(-1)^{\zeta}} \times (\text{GUE correlator, } e^{S_0}) \tag{4.40}$$

One can interpret this rules as arising from an anomaly in $(-1)^{\mathsf{F}}$. This arises for example in any theory with an odd number of Majorana fermions. The Hilbert space is a representation of the Clifford algebra and there is no chirality in odd dimensions. From the path integral perspective there is an odd number of fermion zero-modes which cannot be soaked by interactions which would involve an even number of fermions. The factor of $\sqrt{2}$ relating path integral to traces can be inferred by considering N free fermions (the path integral is $2^{N/2}$ while the dimension of the Hilbert space is $2^{(N-1)/2}$ if N is odd.)

4.3 "Unorientable" JT gravity

In the BF description, unorientable JT gravity can be obtained by replacing the gauge group $\mathrm{PSL}(2,\mathbb{R})$ by its double cover $\mathrm{PGL}(2,\mathbb{R})$. This is the group of 2×2 invertible real matrices modulo multiplication by a nonzero real scalar. This group includes the element

$$U_{\mathsf{R}} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}, \tag{4.41}$$

which $\mathrm{PSL}(2,\mathbb{R})$ would not. Other than this the groups are the same; multiplication by a nonzero real scalar can be used to set the determinant of any matrix to be either 1, meaning its in $\mathrm{PSL}(2,\mathbb{R})$, or -1, meaning its the product of a matrix in $\mathrm{PSL}(2,\mathbb{R})$ times U_{R}. (This double cover does not care about spin structure since I and $-I$ are still identified in $\mathrm{PGL}(2,\mathbb{R})$.)

What is the geometric interpretation of U_{R}? Whenever a holonomy includes U_{R} the meaning is that there has been an orientation reversal. For example, in the case of the cylinder the "holonomy" from one boundary relative to the other could be $U_\varrho = \mathrm{diag}(e^{\varrho/2}, e^{-\varrho/2})$ (standard choice) or $U_{\mathsf{R}}{\cdot}U_\varrho = \mathrm{diag}(e^{\varrho/2}, -e^{-\varrho/2})$ which involves an orientation reversal prior to gluing. This is referred to in [47] as the twisted trumpet.

Describe first orientable surfaces. Since we are gauging orientation reversal, the orientation of a given boundary is not meaningful. But once we choose an orientation in one boundary, the relative orientation of the other $n-1$ boundaries will be meaningful since they can be measure by "holonomies" obtained from parallel transport between boundaries. This generates 2^{n-1} topologically distinct contributions. The partition function in any of them is exactly the same as the one we computed in oriented (bosonic) JT gravity. This therefore produces a simple factor

$$\text{(Path integral over oriented surface)} = 2^{n-1} \cdot Z^{\mathrm{JT}}_{g,n}. \tag{4.42}$$

It is easy to check that, for genus zero, the loop equations of the $\beta = 1$ and $\beta = 2$ resolvants are related by

$$R^{\beta=1}_{0,n}(x_1, \ldots, x_n) = 2^{n-1} R^{\beta=2}_{0,n}(x_1, \ldots, x_n) \tag{4.43}$$

Loop equations for Dyson ensembles

Show this. To remind you, the loop equations of the Dyson ensembles are given by $2y(x)R_g(x,I) + F_g(x,I) \sim 0$ where

$$\begin{aligned} F_g(x,I) \,&=\, \Big(1-\frac{2}{\beta}\Big)\partial_x R_{g-\frac{1}{2}}(x,I) + R_{g-1}(x,x,I) + \sum_{\text{stable}} R_h(x,J)R_{g-h}(x,I/J) \\ &\quad +2\sum_{k=1}^{n}\Big(R_0(x,x_k) + \frac{1}{\beta}\frac{1}{(x-x_k)^2}\Big)R_g(x,I/x_k). \end{aligned} \tag{4.44}$$

where $R(x) = \sum_j (x-\lambda_j)^{-1}$. In the case of GSE it does not include the two-fold degeneracy.

Orientable surfaces are made out of three-holed spheres. To build a hyperbolic metric on an arbitrary possibly unorientable Y, we need another kind of building block. This is obtained from a three-holed sphere Y_0 by replacing one or more of its boundary circles with crosscaps. If χ is even we can make unorientable surfaces only out of orientation-reversal when gluing.

We can construct another non-trivial theory if we insert a sign $(-1)^{n_c}$. This relates the GOE ensemble with the GSE. To see this, notice that if we write

$$\widetilde{R}_g(I;L) = 2^{|I|} R_g(I;L/2), \tag{4.45}$$

then the loop equation for the GSE is identical to the GOE one with the sign of the crosscap reversed. This is the actual observable that $\mathrm{Tr}(x-H)^{-1}$ would compute which includes the degeneracy. The rescaling $L \to L/2$ just guarantees the disk amplitudes are the same.

GOE vs GSE

Use the loop equations to show this.

How to compute the path integral on unorientable surfaces? At this point we are forced to use the torsion; JT gravity is not tree-level exact anymore on unorientable surfaces!

Describe cross-cap torsion calculation. We can represent the crosscap by a cylinder with two circle boundaries S and S'. On S' we identify antipodal points making it an internal circle now. If we denote by $U = \mathrm{diag}(e^{b/2}, e^{-b/2})$ the holonomy around S then the holonomy around S', which we call V, should satisfy $U = V^2$. Since going around S' involves an orientation reversal we take $V = U_{\mathsf{R}} \cdot \mathrm{diag}(e^{b/4}, e^{-b/4}) = \mathrm{diag}(e^{b/4}, -e^{-b/4})$. By an argument similar to the one that applied to the circle, the torsion of the crosscap involves the determinant of the operator $s \to VsV^{-1} - s$, with eigenvalues $-1 - e^{\pm b/2}$ and a careful treatment of the zero-modes. The result is

$$\tau_C = 2\cosh^2 \frac{b}{4} \cdot \mathrm{d}b \cdot (\mathrm{d}\varrho)^{-1}. \tag{4.46}$$

We can now derive a measure for integration over the moduli space of hyperbolic surfaces which are possibly unorientable. They can be built out of three-holed spheres with orientation reversing gluing (this would be the trivial case) but they can also involve three-holed-sphere where either one or two boundaries are replaced by crosscaps. Using the gluing rules described in the previous lecture one automatically gets

$$\tau_Y = \prod_{i=1}^{3g-3} \mathrm{d}b_i \, \mathrm{d}\varrho_i \prod_{\alpha=1}^{n} \frac{1}{2} \coth \frac{b_\alpha}{4} \, \mathrm{d}b_\alpha. \tag{4.47}$$

Here b_α are the lengths of the crosscap geodesics. The twist cancels in the gluing, as it should. Precisely this measure was previously "bootstrapped" by Norbury, with the assumption that it takes a factorized form and that its independent of choice of coordinates. This is consistent with the gravity calculation which should be invariant under change of coordinates (the factorized form is not obvious although very reasonable given the gluing property of the torsion).

We can perform another small check; compute the crosscap with one boundary. Recalling the trumpet with torsion boundary conditions, and the gluing relations for the torsion, leads to

$$Z_{\frac{1}{2},1}(\beta) = \int_0^\infty \mathrm{d}b \, Z_{\mathrm{JT}}^{\mathrm{trumpet}}(\beta, b) \, \frac{1}{2} \coth\left(\frac{b}{4}\right). \tag{4.48}$$

Even though this is divergent, the integral can be succesfully compared to the prediction from loop equations. This was actually shown to all orders recently by Stanford [63].

4.4 "Unorientable" fermionic JT gravity

We now want to sum over both unorientable surfaces, and over spin structures. In the BF description we included orientable surfaces by replacing $\mathrm{PSL}(2,\mathbb{R})$ by $\mathrm{PGL}(2,\mathbb{R})$, and we included a spin structure by replacing $\mathrm{PSL}(2,\mathbb{R})$ by $\mathrm{SL}(2,\mathbb{R})$. To do both things we need to take the appropriate double cover of $\mathrm{SL}(2,\mathbb{R})$. There are two such groups which we can call pin^+ and pin^-:

For pin^+ one wants the group of real matrices with determinant ± 1 meaning one again includes $\mathrm{diag}(1,-1)$. This means that $\mathsf{R}^2 = 1$. The CPT theorem implies that any field theory on a Lorentzian spacetime has a non-unitary symmetry RT which satisfies $\mathsf{RT}^2 = (-1)^{\mathsf{F}}$. Therefore the fact that $\mathsf{R}^2 = 1$ implies that on the boundary $\mathsf{T}^2 = (-1)^{\mathsf{F}}$, up to possible anomalies.

For pin^- we work with the group of matrices of determinant 1 that are either real or imaginary, implying one includes the element $\mathsf{R} = \mathrm{diag}(\mathrm{i}, -\mathrm{i})$. In this case $\mathsf{R}^2 = -1$ which acts trivially on bosonic fields and produces a negative sign on fermions. This implies that it acts as $\mathsf{R}^2 = (-1)^{\mathsf{F}}$. The CPT theorem now implies that time reversal acts on the boundary as $\mathsf{T}^2 = 1$ up to possible anomalies.

The procedure should be now clear. For a given orientation and spin structure we can perform the JT gravity path integral obtaining the result above. The non-trivial ingredient now is the possibility to incorporate a sum over $\mathrm{pin}^{\pm}$ structures together with possible TQFT weighting them in different ways. We conclude by summarizing the possibilities:

Sum over pin^- structures: The topological invariant on a surfaces with pin^- structure that generalizes the mod 2 TQFT is given by

$$\exp(-\mathrm{i}\pi\eta/2), \tag{4.49}$$

where η is the Atiyah-Patodi-Singer eta invariant of the self-adjoint operator $\mathrm{i}\not{D}$. For any manifold this phase is the eighth-root of unity giving essentially 8 possible theories where

$$Z_{\mathrm{JT}} \cdot \sum_{\mathrm{pin}^-} \exp(-\mathrm{i}\pi N\eta/2), \tag{4.50}$$

and N is defined mod 8. On orientable surfaces the eta invariant reduces to the mod 2 invariant and we recover the two theories analyzed earlier. Sums over pin structures can be used combining this fact with the locality of the eta-invariant, and the result for a crosscap.

Let us work out one example to illustrate how it works. Take the TQFT with $N = 0, 4$ mod 8. One can show

$$\sum_{\mathrm{pin}^-} \exp(-\mathrm{i}\pi 0\eta/2) = 2^{2g+n-2}(1 + (-1)^{n_{\mathrm{R}}}), \tag{4.51}$$

$$\sum_{\mathrm{pin}^-} \exp(-\mathrm{i}\pi 4\eta/2) = 2^{2g+n-2}(-1)^{n_{\mathrm{c}}}(1 + (-1)^{n_{\mathrm{R}}}). \tag{4.52}$$

Multiplying this by the unoriented JT gravity partition function, which we already determined is dual to the GOE/GSE ensemble, leads to the two matrix ensembles

$$N = 0 \text{ mod } 8 \quad \Rightarrow \quad H = \begin{pmatrix} \text{GOE}_1 & 0 \\ 0 & \text{GOE}_2 \end{pmatrix}, \tag{4.53}$$

$$N = 4 \text{ mod } 8 \quad \Rightarrow \quad H = \begin{pmatrix} \text{GSE}_1 & 0 \\ 0 & \text{GSE}_2 \end{pmatrix}. \tag{4.54}$$

Therefore the case $N = 0$ mod 8 corresponds to a theory where quantum mechanically there is a $\mathbb{Z}_2^{\mathsf{T}} \times \mathbb{Z}_2^{\mathsf{F}}$ group generated by $\mathsf{T}^2 = 1$ and $(-1)^{\mathsf{F}}$. When $N = 4$ mod 8 the only difference is that now $\mathsf{T}^2 = -1$. Similarly, all choices of N can be mapped to possible anomalies in the classical $\mathbb{Z}_2^{\mathsf{T}} \times \mathbb{Z}_2^{\mathsf{F}}$ symmetry.

This case is important because when we consider its generalization to $\mathcal{N} = 1$ supergravity it will generate the rest of the Altland-Zirnbauer ensembles of random matrices [47].

Sum over pin$^+$ structures: In this case we need to analyze what invariants we can define on unorientable pin$^+$ structures. This is actually a mod 2 index of the Dirac operator (non-chiral since we include unorientable surfaces). This is sometimes denoted by

$$(-1)^{\widetilde{\zeta}}. \tag{4.55}$$

Therefore there are two theories corresponding to pin plus structures.

When we sum over pin structures without the mod 2 TQFT, this corresponds to a non-anomalous boundary theory with both T and $(-1)^{\mathsf{F}}$ but

$$\mathsf{T}^2 = (-1)^{\mathsf{F}}. \tag{4.56}$$

making the group $\mathbb{Z}_4^{\mathsf{T}}$ where the super-index indicates it involves time reversal. This means that the Hilbert space decomposed into two sectors. The bosonic sector leads to the GOE ensemble and the fermionic sector to the GSE ensemble. This can be verified using the relation

$$\sum_{\text{pin}^+} 1 = 2^{2g+n-2}(1 + (-1)^{n_{\text{R}}+n_{\text{c}}}). \tag{4.57}$$

In the presence of the mod 2 TQFT we expect an anomaly. There is only one possibility which can be written as

$$\mathsf{T}^2 = \mathrm{i}(-1)^{\mathsf{F}}. \tag{4.58}$$

The anomaly can be moved around (for example can be put in the fermion parity which would then square to minus one) but cannot be removed. Importantly the antiunitarity of T implies

$$\mathsf{T}(-1)^{\mathsf{F}} = -(-1)^{\mathsf{F}}\mathsf{T}. \tag{4.59}$$

Therefore time-reversal exchanges the bosonic and fermionic blocks and does nothing otherwise. This is consistent with

$$\sum_{\mathrm{pin}^+}(-1)^{\widetilde{\zeta}} = 2^{2g+n-1}\,\delta_{n_{\mathrm{R}},0}\,\delta_{n_{\mathrm{c}},0}. \tag{4.60}$$

We see the contribution with crosscaps vanish identically after summing over pin structures, confirming that the GUE ensemble is the relevant one here.

4.5 JT gravity with end-of-the-world branes

An interesting generalization, with important implications to questions such as reproducing the Page curve from gravity, involves the addition of end-of-the-world branes to JT gravity. This was derived by Penington, Shenker, Stanford and Yang [64].

Another generalization I won't have time to cover is JT gravity with propagating matter.

5 Lecture 4: Supergravity

5.1 $\mathcal{N}=1$ JT gravity

5.1.1 Basics

As should be clear by now, the most efficient way to describe a generalization of JT gravity with one supercharge is through the BF description. We replace the group $\mathrm{SL}(2,\mathbb{R})$ by its smallest supersymmetric generalization

$$\mathrm{SL}(2,\mathbb{R}) \to \mathrm{OSp}'(1|2) = \mathrm{OSp}(1|2)/\mathbb{Z}_2. \tag{5.1}$$

This is the group of linear transformations of two bosonic variables u, v and one fermionic θ that preserves the symplectic form

$$\omega = \mathrm{d}u\,\mathrm{d}v + \frac{1}{2}\mathrm{d}\theta^2. \tag{5.2}$$

We are modding out by the transformation $u, v|\theta \to -u, -v| - \theta$ which commutes with everything. The bosonic generators can be written as

$$\mathsf{e} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad \mathsf{f} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad \mathsf{h} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \tag{5.3}$$

and the fermionic ones as

$$\mathsf{q}_1 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & -1 & 0 \end{array}\right), \quad \mathsf{q}_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right). \tag{5.4}$$

Supergravity

Write explicitly the action of $\mathcal{N}=1$ JT supergravity using the BF formulation. Remember to replace trace by the supertrace, which for a given supermatrix $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is given by $\mathrm{STr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, D$.

The JT gravity path integral will naturally localize into flat connections, modulo gauge transformations. This space again has multiple components and the connection with gravity forces us to restrict to one where all holonomies are hyperbolic (after reducing modulo odd variables). We can refer to this as a $\mathcal{N}=1$ generalization of Teichmuller space.

The geometric information is encoded in holonomies in the following way

$$U = \pm \left(\begin{array}{cc|c} e^{b/2} & 0 & 0 \\ 0 & e^{-b/2} & 0 \\ \hline 0 & 0 & \delta \end{array}\right), \tag{5.5}$$

where b is the geodesic length and $\delta = \pm 1$ denote the spin structure. The overall sign is not meaningful in $\mathrm{OSp}'(1|2)$.

The first step, as we did for JT gravity, is to study the disk and the cylinder. The path integral over the disk reduces to a straightforward generalization of the Schwarzian action with one supercharge, defined in [65]. The boundary mode is parameterized by super-reparametrizations of the $\mathcal{N}=1$ super-line (τ, θ) that includes both a bosonic component $\tau \to f(\tau)$ as well as fermionic $\theta \to \theta + \eta(\tau)$. The action is

$$I = -\Phi_r \int \mathrm{d}\tau \, \{\tan \frac{\pi f}{\beta}, \tau\} + \eta\, \eta''' + 3\eta'\eta'' - \{\tan \frac{\pi f}{\beta}, \tau\}\eta\, \eta'. \tag{5.6}$$

The zero-mode of this action are the generators of the isometries of the $\mathcal{N}=1$ hyperbolic disk. This isometry group is precisely $\mathrm{OSp}(1|2)$ of dimension $3|2$. The fermion zero-modes have a behavior $\eta \sim e^{\pm \mathrm{i}\frac{1}{2}\frac{2\pi\tau}{\beta}}$.

The path integral localizes by a supersymmetric analog of the Duistermaat-Heckman formula [26] and the one-loop determinant is again given by the “rotation angles” of a U(1) symmetry. The one-loop determinant for bosons and fermions are

$$Z_{\text{one-loop}} = \underbrace{\prod_{n\geq 2} \frac{1}{n/(2\beta)}}_{\text{Schwarzian mode}} \cdot \underbrace{\prod_{m\geq 3/2} m/(2\beta)}_{\text{Fermion}} = \sqrt{\frac{2}{\pi\beta}}. \tag{5.7}$$

To derive this result we can use that in zeta-function regularization $\prod_{m\geq 1/2} m/(2\beta) = \sqrt{2}$. This is the partition function of a single Majorana fermion. The final answer for the disk partition function is

$$Z_{\text{disk}} = \sqrt{\frac{2}{\pi\beta}}\, e^{\frac{\pi^2}{\beta}}. \tag{5.8}$$

The different factor of β can be traced back to the fact that the $\mathcal{N}=1$ has two new fermionic isometries, which induces a new factor of β in the numerator. The counting is straightforward.

The spectral curve derived from this formula is

$$\rho(E)=\frac{\sqrt{2}\cosh(2\pi\sqrt{E})}{\pi\sqrt{E}}, \qquad y(x)=-\frac{\sqrt{2}\cos(2\pi\sqrt{-x})}{\sqrt{-x}}. \tag{5.9}$$

A similar calculation on the trumpet gives

$$Z_{\rm SJT}^{\rm trumpet}(\beta,b)=\frac{1}{\sqrt{2\pi\beta}}\,e^{-\frac{b^2}{4\beta}}. \tag{5.10}$$

This has the same power of β since the trumpet has no fermionic isometries[12]. It is multiplied by $\sqrt{2}$ compared to the bosonic answer, the partition function of a Majorana fermion. The cylinder partition function is

$$Z_{0,2}(\beta_1,\beta_2)=2\int b\mathrm{d}b\, Z_{\rm SJT}^{\rm trumpet}(\beta_1,b)Z_{\rm SJT}^{\rm trumpet}(\beta_2,b). \tag{5.11}$$

The factor of 2 arises from the two spin structures "orthogonal" to the gluing geodesic.

5.1.2 Measure over moduli space

We can compute the measure over moduli space using the torsion. Again, we have either a symplectic or torsion approach. Both are applicable but the symplectic approach is not particularly simpler and does not apply on unorientable surfaces; for this reason we will use the torsion again.

To compute the combinatorial torsion we need to determine a supergroup measure. We did that earlier by starting with a nondegenerate quadratic form, from which we derived a measure involving a determinant. The generalization of this to supergroups is the following. First, the quadratic form should involve a supertrace instead of a trace to guarantee it is invariant. Second, we need a generalization of $|\det M|$ for supergroups. This is called the Berezinian and has a general definition but we only need its expression for a supermatrix $M=\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right)$, when either B or C vanish

$$\operatorname{Ber} M=\det A\cdot\frac{1}{\det D}. \tag{5.12}$$

The absolute value is replaced by $\operatorname{Ber}' M=\operatorname{sgn}(\det A)\operatorname{Ber} M$.

The torsion associated to bosonic groups is positive, given that it involves positive measures. This is not the case for supergroups and there is some arbitrariness in assigning signs. Even more, on unorientable surfaces the path integral can even be complex.

[12] One can give a simple BF argument for this. The holonomy around the hole is $\pm\mathrm{diag}(e^{b/2},e^{-b/2},\delta)$. There are no fermionic matrices that commute with this holonomy.

Torsion of a circle Let us begin consider the case of a circle since its simpler and take its holonomy to be $U = \pm\mathrm{diag}(e^{b/2}, e^{-b/2}, \delta)$. The torsion is the Berezinian of the operator $\partial s = UsU^{-1} - s$. A simple calculation leads to the bosonic eigenvalues $e^{\pm b} - 1$ and fermionic $\delta e^{\pm b/2} - 1$. The zero-modes contribute $\mathrm{d}b \cdot (\mathrm{d}\varrho)^{-1}$ for the same reason as in the bosonic theory. The final answer is

$$\tau_S = -\delta(e^{b/4} + \delta e^{-b/4})^2 \mathrm{d}b \cdot (\mathrm{d}\varrho)^{-1} = \begin{cases} 4\sinh^2(\frac{b}{4})\, \mathrm{d}b \cdot (\mathrm{d}\varrho)^{-1} & \text{NS spin structure,} \\ -4\cosh^2(\frac{b}{2})\, \mathrm{d}b \cdot (\mathrm{d}\varrho)^{-1} & \text{R spin structure.} \end{cases} \tag{5.13}$$

Torsion of a three-holed sphere The description of the moduli space can be made very similar to the case of bosonic gravity. We have three holes with holonomy U, V, W constrained by $UVW = 1$ and defined only modulo overall conjugation. This leaves total of $3|2$ moduli. The three bosonic ones are the usual geodesic lengths while the remaining 2 moduli are fermionic.

Take $x \in \mathrm{osp}(1|2)$ and consider the quadratic form $||x||^2 = 2\mathrm{STr}\, x^2 = 4x_h^2 + 4x_e x_f + 8x_1 x_2$. The Berezinian of this metric is one, so the natural measure is $[\mathrm{d}x_e\, \mathrm{d}x_f\, \mathrm{d}x_h | \mathrm{d}x_1\, \mathrm{d}x_2]$. The normalization of the quadratic form is chosen so that it reduces to the one considered in the bosonic case, which we show to coincide with the Weil-Petersson form with the usual normalization. Finally the form on the group manifold in terms of U is the measure on $U^{-1}\mathrm{d}U$.

We parametrize the holonomies by $U = RU_0R^{-1}$ and $V = RV_0R^{-1}$ with

$$U_0 = \delta_1 \left(\begin{array}{cc|c} e^{b_1/2} & \kappa & 0 \\ 0 & e^{-b_1/2} & 0 \\ \hline 0 & 0 & \delta_1 \end{array}\right) \cdot e^{\xi \mathfrak{q}_1}, \qquad V_0 = \delta_2 \left(\begin{array}{cc|c} e^{-b_2/2} & 0 & 0 \\ 1 & e^{b_2/2} & 0 \\ \hline 0 & 0 & \delta_2 \end{array}\right) \cdot e^{\psi \mathfrak{q}_2}. \tag{5.14}$$

We can compute the torsion through $\tau = \mathrm{vol}(U)\, \mathrm{vol}(V)/\mathrm{vol}(R)$. We also need to find an equation that relates κ with b_3. The final answer for the torsion is

$$\tau_Y = \frac{2\sinh\frac{b_1}{2}\sinh\frac{b_2}{2}\sinh\frac{b_3}{2}}{(\delta_1 e^{b_1/2} - 1)(\delta_2 e^{b_2/2} - 1)} [\mathrm{d}b_1\, \mathrm{d}b_2\, \mathrm{d}b_3 | \mathrm{d}\xi \mathrm{d}\psi] \tag{5.15}$$

Supergravity torsion

Reproduce the torsion of $\mathcal{N} = 1$ supergravity.

Measure over moduli space We now can glue the pieces. Consider a closed surface first. We take the product of the torsion of all $2g - 2$ three-holed spheres $t \in Y$ and the product of the inverse torsion of all circles $s \in S$. Since the torsion of the three-holed sphere is not manifestly invariant under permutation of boundaries we need to make an arbitrary choice of labels which we call $(b_1, b_2, b_3) \to (a_t, b_t, c_t)$ where $t \in Y$ is the three-holed sphere under consideration. The final answer after combining all terms is

$$\tau = \frac{1}{2}(-1)^{w_{\rm R}} \prod_{s\in S}[\mathrm{d}b_s \mathrm{d}\varrho_s] \prod_{t\in Y} \frac{1}{4}\delta_{a_t}\delta_{b_t} e^{-(a_t+b_t)/4}(e^{c_t/4} - \delta_{c_t} e^{-c_t/4})[\mathrm{d}\xi_t\, \mathrm{d}\psi_t]. \tag{5.16}$$

The prefactor of 1/2 comes from taking into account the $\mathbb{Z}_2$ symmetry $(-1)^{\mathsf{F}}$. $w_{\rm R}$ is the number of interior Ramond boundaries.

A similar formula can be written in the case of surfaces with boundaries in an obvious way, although the trigonometric factors are not that nice. This formula implies that for $g = 0$ the volume vanish since we have a fermionic integral of a measure independent of fermionic coordinates. This does not happen at higher genus since the mapping class group impose inequalities that might involve fermionic coordinates.

Finally, besides integrating over lengths and twist we also need to sum over spin structures. When performing this sum we are free to insert our mod 2 index TQFT $(-1)^{\zeta}$. This leads to two different $\mathcal{N} = 1$ supergravity theories.

5.1.3 Duality with random matrices

Theory with anomalous $(-1)^{\mathsf{F}}$ In this case $H = Q^2$ and Q is a self-adjoint operator with no further structure since the fermion parity operator is anomalous. Therefore it is reasonable to expect Q to be taken from the GUE ensemble, with the partition function being

$$Z_{\rm gravity}(\beta) = \sqrt{2}\, \mathrm{Tr}\, e^{-\beta Q^2}. \tag{5.17}$$

The spectral curve of the GUE ensemble for Q is therefore

$$Z_{\rm disk}(\beta) = \sqrt{2} \int_{-\infty}^{\infty} \mathrm{d}q\, \rho_0(q)\, e^{-\beta q^2}, \qquad \rho_0(q) = \frac{\cosh(2\pi q)}{\pi}. \tag{5.18}$$

This is a new situation; the spectral curve has support on the whole real axis. This is a double-scaling limit where both ends go to infinity. This is also in the same universality class as the Gross-Witten-Wadia model[13].

Since the model has a spectral curve with support on whole real axis the loop equations predict all higher genus corrections (other than disk and cylinder) to vanish. The reason is that for the GUE ensemble the loop equations get contributions solely from poles at the edges but there are no edges here. This is consistent with the result above if we sum over spin structures without $(-1)^{\zeta}$, one can show this vanishes.

Theory with $(-1)^{\mathsf{F}}$ **symmetry** In this case the Hilbert space should decompose into two sectors of fermionic and bosonic states. This implies that the supercharge now has the following structure

$$Q = \left(\begin{array}{c|c} 0 & \mathsf{Q} \\ \hline \mathsf{Q}^{\dagger} & 0 \end{array}\right), \qquad H = \left(\begin{array}{c|c} \mathsf{Q}\mathsf{Q}^{\dagger} & 0 \\ \hline 0 & \mathsf{Q}^{\dagger}\mathsf{Q} \end{array}\right). \tag{5.19}$$

[13] This was important to resolve some puzzle when considering deformations of supergravity [59].

where $\mathsf{Q} : \mathcal{H}_f \to \mathcal{H}_b$ and $\mathsf{Q}^\dagger : \mathcal{H}_b \to \mathcal{H}_f$ is its adjoint. We can use a singular value decomposition and call λ_j the singular values of Q. This translate to eigenvalues of both $\mathsf{Q}^\dagger\mathsf{Q}$ as well as $\mathsf{Q}\mathsf{Q}^\dagger$. This means that for each eigenstate in the fermionic sector there is an equivalent eigenstate in the bosonic sector. This is also true if $\lambda_j = 0$. The exception is when $\dim\mathcal{H}_b \neq \dim\mathcal{H}_f$ where Q is rectangular and there will be states of zero energy that are protected and appear only in one sector.

This is one of the ten Altland-Zirnbauer ensembles. The measure of integration over the singular values become

$$\prod_i \int \mathrm{d}\lambda_i\, |\lambda_i|^\alpha \prod_{i<j} |\lambda_i^2 - \lambda_j^2|^\beta \; e^{-L\sum_j V(\lambda_j^2)} \tag{5.20}$$

where for our specific case of the complex matrix Q, the coefficients are $(\alpha, \beta) = (1, 2)$.

We can reproduce $\mathcal{N} = 1$ supergravity by choosing a Hilbert space with equal dimension for bosonic and fermionic states. Then the partition function will be

$$\mathrm{Tr}\, e^{-\beta H} = 2\sum_j e^{-\beta\lambda_j}. \tag{5.21}$$

Since the loop equations are normally derived in terms of sums over unequal eigenvalues, we need to multiply the AZ ensemble answers by a factor of $2^{\,\#\text{ of boundaries}}$.

We can verify this for the cylinder. We should get from gravity four times the bosonic answer. Indeed each trumpet has an extra factor of $\sqrt{2}$ from the fermion path integral, while there is an extra 2 from the sum over spin structures of the cylinder making a total factor of 4.

It can be verified by a Laplace transform that the loop equation of the $(\alpha, \beta) = (1, 2)$ AZ ensemble coincide with the topological recursion computing the volumes of moduli space of $\mathcal{N} = 1$ surfaces. Let us mention some intermediate steps. First of all, the matrix model loop equations for the $(\alpha, \beta) = (1, 2)$ ensemble can be written as

$$\begin{aligned} bV_g(b, B) = &-\frac{1}{2}\int_0^\infty (b'\mathrm{d}b')(b''\mathrm{d}b'')\, D(b' + b'', b) \Big(V_{g-1}(b', b'', B) + \sum_{\text{stable}} V_{h_1}(b', B_1)V_{h_2}(b'', B_2)\Big) \\ &-\sum_{k=1}^{|B|}\int_0^\infty b'\mathrm{d}b'\, (D(b' + b_k, b) + D(b' - b_k, b))V_g(b', B/b_k). \end{aligned} \tag{5.22}$$

where

$$D(x, y) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, \frac{e^{-xz}}{2zy(-z^2)}\, \sinh(yz). \tag{5.23}$$

When we insert the spectral curve of $\mathcal{N} = 1$ JT gravity we obtain precisely

$$D(x, y) = \frac{1}{8\pi}\Big(\frac{1}{\cosh\frac{x-y}{4}} - \frac{1}{\cosh\frac{x+y}{4}}\Big). \tag{5.24}$$

It was shown by Stanford and Witten, and independently by Norbury, that the Weil-Petersson volumes of $\mathcal{N} = 1$ surfaces follow precisely this recursion relation with this kernel. This proves that $\mathcal{N}=1$ JT gravity including the mod 2 index is dual to the $(1,2)$ AZ matrix ensemble.

Recursion

Relate the loop equations with a Mirzakhani-looking recursion for any spectral curve. In other words, derive the equation (5.23).

If you recall the derivation of Mirzakhani recursion, it involves computing the length on a boundary circle of a segment where geodesics have certain properties. This singles out a three-holed sphere with no moduli. In the supergravity case the three-holed sphere has fermionic moduli and have to be integrated. This causes the changes in the recursion kernel that matches exactly with the loop equations.

5.1.4 "Unorientable" $\mathcal{N}=1$ supergravity

In the BF language, in order to define unorientable supergravity we need to select an automorphism of the group which is orientation reversal. We will work with the basis of generators we wrote above $\{\mathsf{e},\mathsf{f},\mathsf{h};\mathsf{q}_1,\mathsf{q}_2\}$. If we conjugate the bosonic generators by the orientation reversal $\mathrm{diag}(1,-1)$ we obtain

$$\mathsf{e}\to-\mathsf{e},\qquad \mathsf{f}\to-\mathsf{f},\qquad \mathsf{h}\to\mathsf{h}. \tag{5.25}$$

This preserves the $\mathrm{sl}(2,\mathbb{R})$ algebra $[\mathsf{h},\mathsf{e}]=2\mathsf{e}$, $[\mathsf{h},\mathsf{f}]=-\mathsf{f}$ and $[\mathsf{e},\mathsf{f}]=\mathsf{h}$. What about fermions? We need to guarantee that the action of this transformation on fermions keeps the algebra unchanged. The relations $\mathsf{q}_1^2=-\mathsf{e}$ and $\mathsf{q}_2^2=\mathsf{f}$ imply that the transformation has to be $\mathsf{q}_1\to\pm i\mathsf{q}_1$ and $\mathsf{q}_2\to\pm i\mathsf{q}_2$. On the other hand $\{\mathsf{q}_1,\mathsf{q}_2\}=\mathsf{h}$ implies the two signs have to be opposite so if $\mathsf{q}_1\to\eta i\mathsf{q}_1$ then $\mathsf{q}_2\to-\eta i\mathsf{q}_2$ where $\eta=\pm1$. The only other non-trivial commutators to check are $[\mathsf{e},\mathsf{q}_2]=\mathsf{q}_1$ and $[\mathsf{f},\mathsf{q}_1]=\mathsf{q}_2$.

The conclusion of the previous discussion is that applying an orientation reversal twice leaves the bosonic generators unchanged while it acts on fermions by reversing their sign. This means that such a transformation satisfies $\mathsf{R}^2=(-1)^{\mathsf{F}}$, implying that $\mathcal{N}=1$ supergravity only allows for a sum over pin^- structures.

One can provide a holographically dual argument for the restriction to pin^- structures. The CPT theorem implies that a pin^+ structure leads to a time-reversal symmetry acting classically as $\mathsf{T}^2=(-1)^{\mathsf{F}}$. This cannot happen for a theory with one supercharge; since the supercharge are fermionic they have to come in representations of T which are at least two-dimensional.

We can therefore define 8 different unorientable $\mathcal{N}=1$ JT gravity by including the eta-invariant TQFT. These theories saturate the remainder of the ten Altland-Zirnbauer random matrix ensembles.

5.2 $\mathcal{N}=2$ JT gravity

The previous derivation was extended to $\mathcal{N}=2$ JT supergravity in [66]. I will only have time to cover some main aspects of the calculation.

5.2.1 Warmup: JT gravity coupled to a gauge field

To be added

5.2.2 Derivation of the matrix ensemble

In this case we will begin by analyzing the random matrix ensemble. This has not been done before [66] for systems with extended supersymmetry.

$\mathcal{N}=2$ quantum mechanics implies the existence of two charges Q and $Q^\dagger$ that satisfy the algebra

$$Q^2 = Q^{\dagger 2} = 0, \qquad \{Q, Q^\dagger\} = H. \tag{5.26}$$

We also assume the existence of an R-symetry U(1) generated by J satisfying

$$[J, Q] = Q, \quad [J, H] = 0. \tag{5.27}$$

The operator distinguishing bosons and fermions is now given by

$$(-1)^{\mathsf{F}} = e^{\pm \mathrm{i}\pi J}. \tag{5.28}$$

The Hilbert space decomposes according to the spectrum of R-charges $\mathcal{H} = \oplus_k \mathcal{H}_k$. The supercharge decomposes accordingly

$$Q = \sum_k Q_k, \qquad Q_k : \mathcal{H}_k \to \mathcal{H}_{k+1}, \tag{5.29}$$

and the algebra has two types of irreducible multiplets. BPS multiplets are invariant under all supercharges $Q\psi_k = Q^\dagger \psi_k = 0$ and come as one state of charge k. Non-BPS multiplets come in pairs of charge $(k, k+1)$ and $Q\psi_k \sim \psi_{k+1}, Q^\dagger \psi_{k+1} \sim \psi_k$. The Hilbert space therefore further decomposes into

$$\mathcal{H}_k = \underbrace{\mathcal{H}_k^0}_{\text{BPS states of charge k}} \oplus \underbrace{\mathcal{H}_k^+}_{\text{From multiplet } (k,k+1)} \oplus \underbrace{\mathcal{H}_k^-}_{\text{From multiplet } (k,k-1)} \tag{5.30}$$

We would like to construct a random matrix model where we integrate over all supercharges satisfying the right algebra, and such that the ensemble is invariant under unitaries acting on each subsector of charge k. This leads to two immediate issues:

- A naive prescription is to integrate over all Q_k as if they were complex matrices. This is wrong since the algebra $Q^2 = 0$ imply that

  $$Q_k \cdot Q_{k-1} = 0. \tag{5.31}$$

  This has to be supplemented as a constraint.

- Consider the symmetry group $\mathrm{U}(L_k)$ acting on $\mathcal{H}_k$. Both matrices Q_{k-1} and Q_k are affected, via left- or right-multiplication, by this unitary transformation. Therefore in the reduction to its singular value integral, the measure one derives (analogous to the Vandermonde term) will not obviously factorize between different supermultiplets.

To say it in another words, we will to compute the measure factor

$$\int \prod_k \mathrm{d}Q_k \prod_s \delta(Q_{s+1} \cdot Q_s) \tag{5.32}$$

to a suitable integral over "eigenvalues" and determine the measure. This calculation was done in [66] with the result

- The $(k, k+1)$ multiplets are statistically independent of each other.
- The measure over each supermultiplet reduces to that of the Altland-Zirnbauer ensemble with

  $$\prod_i \int \mathrm{d}\lambda_i\, |\lambda_i|^{\alpha} \prod_{i<j} |\lambda_i^2 - \lambda_j^2|^{\beta} \qquad \Rightarrow \qquad (\alpha, \beta) = (1 + 2L_k^0 + 2L_{k+1}^0, 2). \tag{5.33}$$

  In models where $L_k^0 \sim L$ the BPS contribution to the measure can be absorbed in the matrix potential leading to effectively a $(1,2)$ model (matrix models with logarithmic potentials are also called Penner models). In such cases the information of the BPS states is in the spectral curve, not in the loop equations.
- The wavefunction of supersymmetric states of R-charge k inside $\mathcal{H}_k$ are random.

The result is familiar but yet surprising. It relies on a non-trivial cancellation between the effects enumerated above that want to correlate the singular values of different Q_k's.

5.2.3 Supergravity path integral

In the BF formalism we can define $\mathcal{N} = 2$ JT gravity as a gauge theory with the supergroup $\mathrm{SU}(1,1|1) = \mathrm{OSp}(2|2)/\mathbb{Z}_2$. This is the group of linear transformations acting on a space of dimension $2|1$ preserving an inner product. The maximal bosonic subgroup includes $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{U}(1)$. This implies that the theory includes bosonic JT gravity, a two-dimensional Maxwell field, together with a complex gravitini and dilatini. The path integral localizes to flat $\mathrm{SU}(1,1|1)$ connections which in the appropriate component reduce to $\mathcal{N} = 2$ hyperbolic surfaces.

The holonomy around a geodesic is conjugated to[14]

$$U = e^{\mathrm{i}\phi} \left(\begin{array}{cc|c} -e^{b/2} & 0 & 0 \\ 0 & -e^{-b/2} & 0 \\ \hline 0 & 0 & e^{\mathrm{i}\phi} \end{array} \right). \tag{5.34}$$

b represents the geodesic length while ϕ labels the $\mathrm{U}(1)$ holonomy around it. We have fixed conventions where $\phi = 0$ corresponds to antiperiodic fermions. Periodic fermions can be achieved by setting $\phi = \pi$, what would be referred to as spectral flow in the context of 2d CFT.

[14] One can also work with a $\widehat{q}$-fold cover of $\mathrm{SU}(1,1|1)$ where $\phi \sim \phi + 2\pi/\widehat{q}$ and $\widehat{q}$ is an odd integer. This generalization is important, particularly in the context of SYK [65], but will not make any drastic change in the discussion here.

Something new that happens in this case is that we can add topological theories that affect the disk. This is due to the fact that even at the disk level there is a sum over topologically inequivalent configurations of the U(1) gauge field labeled by their first Chern class. One is free to add a term analog of the theta angle in four dimensions. We call this δ.

The disk partition function is given by

$$Z_{\text{disk}}(\beta,\alpha) = e^{S_0} \sum_{n\in\mathbb{Z}} \exp\left(2\pi \mathrm{i} n\delta\right) \frac{\cos(\frac{\alpha}{2}+\pi n)}{2\pi^3(1-4(\alpha/2\pi+n)^2)}\, e^{\frac{\pi^2}{\beta}(1-4(\alpha/2\pi+n)^2}. \tag{5.35}$$

The prefactor arises from the one-loop determinant. Since the fermions are now charged under the U(1) symmetry, the one-loop determinant depends non-trivially on the U(1) chemical potential.

The spectrum derived from this partition function takes the form

$$\begin{aligned} Z(\beta,\alpha) = & \sum_{k\in\mathbb{Z}+\delta} e^{\mathrm{i}\alpha k} \underbrace{\frac{e^{S_0}\cos(\pi k)}{4\pi^2}}_{\text{BPS states of charge k}} \\ & + \sum_{q\in\mathbb{Z}+\delta-\frac{1}{2}} (e^{\mathrm{i}\alpha(q-\frac{1}{2})} + e^{\mathrm{i}\alpha(q+\frac{1}{2})}) \int_{E_0(q)}^{\infty} \mathrm{d}E\, e^{-\beta E} \underbrace{\frac{e^{S_0}\sinh(2\pi\sqrt{E-E_0(q)})}{8\pi^3 E}}_{\text{Spectrum of non-BPS multiplets}}. \end{aligned} \tag{5.36}$$

where $E_0(q) = q^2/4$. This is the threshold energy of a non-BPS multiplet with average R-charge q, or equivalently with R-charges $k = q \pm 1/2$. We see that the effect of the theta angle δ is simply to add a background charge. This represents a mixed anomaly when $\delta = 1/2$ between charge-conjugation and U(1) symmetry.

We see almost all multiplets have a non-vanishing gap $E_0 \neq 0$ and a square-root edge. This is consistent with the $(1,2)$ ensemble as long as $E_0 \neq 0$. When $q = 0$ is in the spectrum, its gap vanishes and now we get a $1/\sqrt{E}$, again consistent with an $(1,2)$ ensemble.

The cylinder path integral can be obtained from gluing two double-trumpets. The twist parameter comes now with a U(1) partner; the SU$(1,1|1)$ holonomies describing parallel transport from one boundary to another, represented by matrices that commute with U are proportional to diag$(e^{\varrho/2}, e^{-\varrho/2}, -e^{\mathrm{i}\varphi})$. The trumpet partition function is

$$Z_{\text{trumpet}} = \sum_{n\in\mathbb{Z}} \exp\left(2\pi \mathrm{i} n\delta\right) \frac{\cos(\frac{\alpha}{2}+\pi n)}{\pi\beta} e^{-\frac{b^2}{4\beta}-\frac{4\pi^2}{\beta}(\frac{\alpha-\phi}{2\pi}+n)^2}. \tag{5.37}$$

The final answer for the double-trumpet is

$$\begin{aligned} Z &= (2\pi)\int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} b\, \mathrm{d}b Z_{\text{trumpet}}(\beta_1,\alpha_1;b,\phi) Z_{\text{trumpet}}(\beta_2,\alpha_2;b,-\phi) \\ &= \sum_q (e^{\mathrm{i}\alpha_1(q-\frac{1}{2})} + e^{\mathrm{i}\alpha_1(q+\frac{1}{2})})(e^{\mathrm{i}\alpha_2(q-\frac{1}{2})} + e^{\mathrm{i}\alpha_2(q+\frac{1}{2})}) \frac{\sqrt{\beta_1\beta_2}}{2\pi(\beta_1+\beta_2)} e^{-\beta_1 E_0(q) - \beta_2 E_0(q)} \end{aligned} \tag{5.38}$$

This is consistent with the ensemble derived in [66] which predicts the supermultiplets should be statistically independent; notice there is a single sum over multiplets q instead of two. This is achieved concretely by the integral over intermediate U(1) holonomies along the internal circle.

More complicated surfaces can again be built out of three-holed spheres glued to trumpets. The obvious next step is to evaluate the torsion. At this point one can apply the same approach we did so far to the group $\mathrm{SU}(1,1|1)$. This is easier said than done, and multiple subtleties arise that need to be taken care of. These can be read in [66]. Instead we will just point out some salient features of some results.

For example, in $\mathcal{N}=1$ supergravity we argued that all genus zero volumes vanish due to the presence of fermionic moduli. Is this the case in $\mathcal{N}=2$ supergravity? The answer is no. The path integral on the three-holed sphere for example is given by

$$V_{0,3} = -\frac{1}{2\pi}\frac{1}{4}\delta''(\phi_1+\phi_2+\phi_3). \tag{5.39}$$

The first feature we see is the presence of the delta function imposing $\phi_1+\phi_2+\phi_3=0$. This can be understood directly from considering the U(1) gauge field; there are no flat connections unless this constraint is satisfied. The constraint is still satisfied in supergravity but it gets fermionic corrections where now

$$\phi_1+\phi_2+\phi_3 = (\text{fermions}). \tag{5.40}$$

The precise form of the fermion terms can be derived from the condition $UVW=1$ on the $\mathrm{SU}(1,1|1)$ holonomies. Now when we integrate over the fermionic moduli, non-vanishing terms can be picked up from the fermionic terms of the constraint.

To give an example consider the integral

$$\begin{aligned}\int \mathrm{d}^2\theta\, \delta(x+\theta_1\theta_2) &= \int \mathrm{d}^2\theta\, \left(\delta(x)+\delta'(x)\theta_1\theta_2\right),\\ &= \delta'(x)\end{aligned} \tag{5.41}$$

This type of effect leads to the derivatives acting on the delta function.

It is convenient to present these volumes in terms of their Fourier transforms

$$V_{g,n}(q) = (2\pi)^{2-2g}\int \frac{\mathrm{d}^n\phi}{(2\pi)^n} e^{\mathrm{i}q\sum_j \phi_j} V_{g,n}. \tag{5.42}$$

One can show that these volumes are polynomials in both b and q

$$V_{g,n}(q) = \sum_{m=1}^{2g-2+n} (q^2/4)^m v_{g,n,m}(b_1,\ldots,b_n). \tag{5.43}$$

There are a few properties one can easily show by deriving a supersymmetric generaliation of Mirzakhani recursion.

- The volumes are polynomials in q^2 of highest degree equal to $2g-2+n$.

- The term with the highest power of q is equal to the bosonic volumes computed by Mirzakhani.
- As we decrease the power of q^2 by s units, the coefficient $v_{g,n,m=2g-2+n-s}$ is a polynomial in b^2 of degree $3g-3+n-s$. In particular $v_{g,n,0}$ is a polynomial of degree $g-1$. This is also true for $g=0$ since $v_{0,n,0}=0$ vanishes.
- Some examples one can easily evaluate from the recursion:

$$\begin{aligned} V_{0,3} &= \frac{q^2}{4}1, \quad V_{1,1} = \frac{q^2}{4}\frac{b^2+4\pi^2}{48} + \Big(-\frac{1}{8}\Big), \\ V_{0,4} &= \frac{q^4}{16}\frac{4\pi^2+b_1^2+b_2^2+b_3^2+b_4^2}{2} + \frac{q^2}{4}(-3). \end{aligned} \tag{5.44}$$

We can repeat the derivation of Mirzakhani taking into account the fermionic and bosonic extra moduli involved in the three-holed sphere that is used to glue. This produces yet a new set of kernels and applying a Laplace transform similar to Eynard-Orantin, one can show that the gravitational path integral of $\mathcal{N}=2$ JT supergravity is dual to the random matrix ensemble described earlier. The supermultiplets are statistically independent and the spectral curve is

$$y(x) = \frac{\sin(2\pi\sqrt{-x+q^2/4})}{8\pi^2 x} \tag{5.45}$$

5.2.4 "Unorientable" $\mathcal{N}=2$ supergravity

There are two types of unorientable models one can consider.

Type A When including time reversal T we assume that it commutes with the U(1) generators. Notice that this is the convention used in particle theory although is a bit unnatural (would be more reasonable to call this CT since naturally time reversal also flips the charges). In the dual QM this acts within each supermultiplet and leads to the $(0,1)$ and $(3,4)$ ensembles.

Type B The theory has a anti-unitary symmetry CT that anti-commutes with the R-symmetry generator. This relates $Q_k \leftrightarrow Q_{-k-1}$ and otherwise puts no further constraint except for $Q_{-1/2}$ when $\delta=1/2$. This case leads to $(1,1)$ or $(1,4)$ ensembles.

To study the gravity problem requires requires utilizing the CPT theorem to infer how R acts in each case (giving the two choices above) and then consider the possible topological theories that can be added (leading to the two subcases). Some evidence was given in [66] but no general proof of the dualities.

Finally one could consider a case with separate C and T invariant and enumerate the possibilities.

Open question: extend the dualities to $\mathcal{N}=4$ supergravity. This is highly non-trivial for multiple reasons.

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